Properties and applications of Bernoulli random fields with strong dependency graphs

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Revision notes

This is a revision of the thesis I defended in march 2012 at the TU Graz optimized for screen reading. You can find a version optimized for printing on my homepage.

This revision only fixes grammar, spelling, typos, missing indices and improves the layout of some formulae. The mathematical content is the same as in the officially submitted version. Some parts are erroneous the way they have been originally written:

- The proof of proposition 127 is wrong. A loss of control at the boundary of $\mathcal{R}_{\mathcal{P}}$ can be shown in some cases, though. A loss of control at the boundary of can be shown in some cases, though. See the example in [Tem]. I think that a weaker statement, showing such a loss of uniform control over reduced correlations, could be shown. As such a control is only sufficient for analyticity, the statement of proposition 127 seems to optimistic to me, currently.
- Conjecture 63 is wrong. A slightly weaker version is proven if one shows an infinite version of [Ped12][theorem 1.10].
- The optimality in section 5.6.6 is only true if one adds a global orientation to the tree. In this case it is a special case of an infinite version of [Ped12][theorem 1.10].
- The reduced correlations in chapters 5 and 6 are inverses of the reduced correlations as understood in the mainstream mathematical physics literature. Thus, for translation, the resulting bounds should be inverted.

This includes the only error I have found so far, namely.

Feel free to point out any errors or omissions to me and to ask me questions about this thesis. Enjoy reading.

cTc

Abstract

A Bernoulli random field (short BRF) is a collection of $\{0, 1\}$-valued random variables indexed by the vertices of a graph. We investigate BRFs with prescribed marginal parameters and a dependency structure encoded by a graph. A prominent example is Shearer's measure, derived as the extreme case of the Lovász Local Lemma (short LLL). In the case of a finite graph it is constructed from the weighted independent set polynomial of this graph, with weights derived from the prescribed marginal parameters. The LLL is a classic sufficient condition for the existence of Shearer's measure.

The first part of this thesis recapitulates the properties of Shearer's measure, in particular its minimality for certain conditional probabilities of a large class of BRFs. This minimality, specialized to k-independent BRFs indexed by the integers, lets us determine critical probabilities for k-independent homogeneous percolation on trees. The critical probabilities are smooth functions of the branching number of the tree. Furthermore, the minimality allows to characterize the uniform stochastic domination of Bernoulli product fields by the above class of Bernoulli random fields through the existence of Shearer's measure alone. Thus the LLL also yields sufficient condition for this uniform stochastic domination problem.

The second part of this thesis deals with a second BRF linked intimately to the weighted independent set polynomial of a finite graph. It is the Boltzmann measure of the model of a hardcore lattice gas. A classic question is to find estimates of the domain of absolute and uniform convergence and analyticity of the free energy of the above model. We extend cluster expansion techniques to derive improved estimates in the spirit of Dobrushin's condition. The link with the weighted independent set polynomial allows a straightforward interpretation of these estimates as improvements of the LLL. We conclude with a series of specializations and improvements aimed at improving estimates for regular, transitive, grid-like graphs. These graphs are of particular relevance in statistical mechanics.

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Chapter 1

Introduction

1.1 Overview

This thesis treats properties and applications of certain Bernoulli random fields subject to independence conditions encoded by a graph. A *Bernoulli random field* (short BRF) is a collection of Bernoulli random variables $Y := (Y_v)_{v\in V}$ indexed by the vertices of a graph $G := (V, E)$. The prototypical example of such an independence condition is *strong independence with respect to* G, that is non-adjacent subfields of Y are independent:

$$\forall\, U, W \subsetneq V : \quad d(U, W) > 1 \;\Rightarrow\; Y_U := (Y_v)_{v\in U} \text{ is independent of } Y_W\,. \tag{1.1}$$

In this context G is also called a *strong dependency graph* of Y [LSS97, SS05, AS08]. Recall that a subset $W \subseteq V$ is *independent* iff it contains no neighbouring vertices. That is, the *induced subgraph* $G(W)$ of an independent set W is a collection of *isolated vertices*. If Y has strong dependency graph G, then independent sets of G index product subfields of Y. A classic tool to investigate independent sets of a finite graph G is the *independent set polynomial* [HL94]:

$$I_G : \quad \mathbb{C} \to \mathbb{C} \quad z \mapsto \sum_{\text{independent } W\subseteq V} z^{|W|}\,. \tag{1.2}$$

It is the *generating function* of the independent sets of G. We stay for the remainder of this section in the homogeneous and finite setting. There are two particular BRFs strongly related to I_G.

The first BRF is connected to the *Lovász Local Lemma* (short LLL) [EL75], an important tool in the *probabilistic method* in graph theory [AS08]. If $E \neq \emptyset$, then the root of I_G closest to 0 is $p^G_{sh} - 1 \in [-1, 0]$ [SS05]. For $p \in [p^G_{sh}, 1]$ and $q := 1 - p$ define *Shearer's measure* [She85] on $\mathcal{X}_V := \{0,1\}^V$ by

$$\mu_{G,p}(Z = \vec{x}) := \begin{cases} q^{|W|} I_{G(\widehat{W})}(-q) & \text{if } W := V \setminus \operatorname{supp}(\vec{x}) \text{ is independent,} \\ 0 & \text{else,} \end{cases} \tag{1.3}$$

where $\widehat{W} := W \uplus \mathcal{N}(W)$ and $\mathcal{N}(W)$ are the neighbours of W. In particular $\mu_{G,p}(Z = \vec{1}) = I_G(-q)$ and has strong dependency graph G. Let $\mathcal{C}^{\text{strong}}_G(p)$ be the class of all BRFs with strong dependency graph G and marginal parameter p. The importance of Shearer's measure stems from the fact [She85] that for each $p > p_{sh}^G$, every $Y \in \mathcal{C}_G^{\text{strong}}(p)$ and all $U \subseteq W \subseteq V$:

$$\mathbb{P}(Y_W = \vec{1} | Y_U = \vec{1}) \geq \mu_{G,p}(Z_W = \vec{1} | Z_U = \vec{1}) = \frac{I_{G(W)}(-q)}{I_{G(U)}(-q)} > 0\,. \tag{1.4}$$

That is the above conditional probabilities are all well defined and $\mu_{G,p}$ minorates them uniformly. This property allows to reduce questions for large classes of BRFs to Shearer's measure. We apply (1.4) to derive results about percolation and stochastic domination of Bernoulli product fields (short BPF) by BRFs. To this end we investigate the structure, behaviour and representation of Shearer's measure.

The second BRF is well known in statistical mechanics: it is the so-called *hard-core lattice gas*, also known as *animal model* [Dob96b] or *abstract polymer model* [KP86, FP07]. Let $z \in [0, \infty[$ be the *fugacity* or *chemical potential*. For $\vec{x} := (x_v)_{v \in V} \in \mathcal{X}_V$ define its *support* as $\operatorname{supp}(\vec{x}) := \{v \in V : x_v = 1\}$. We define the probability measure $\lambda_{V,z}$ of this model on $\mathcal{X}_V$ by

$$\lambda_{V,z}(\vec{x}) := \begin{cases} \frac{z^{|W|}}{I_G(z)} & \text{if } W := \operatorname{supp}(\vec{x}) \text{ is independent,} \\ 0 & \text{else.} \end{cases} \tag{1.5}$$

In this setting the normalization factor I_G is the so-called *partition function* of the model. This model does not have strong dependency graph G. A key statistic of the model is the *free energy* $-\log(I_G(z))/|V|$ [Rue69, Bax89]. *Cluster expansion* [KP86, Dob96b, MS00] techniques also demand control of the *pinned correlations* $I_{G(W)}(z)/I_G(z)$ for $W \subseteq V$ and complex z. Thus one wants uniform control of the quantity in question in a small complex domain around 0 as one takes the *thermodynamical limit*, that is as $|V| \to \infty$. Extending tree-operator techniques [FP07] we derive weaker sufficient conditions to achieve such uniform control and discuss the relation with analyticity of the free energy in detail.

Seen at a more technical level, the following ratios play a key role in the treatment of the above two themes:

$$\forall\, v \notin W \subseteq V: \quad \frac{|I_{G(W)}(z)|}{|I_{G(W \uplus \{v\})}(z)|}\,. \tag{1.6}$$

Control of these quantities is achieved by finding lower bounds of the form

$$\forall\, v \notin \text{finite } W \subseteq V: \quad \frac{|I_{G(W)}(-r)|}{|I_{G(W \uplus \{v\})}(-r)|} \geq c_{v,W}(r)\,, \tag{1.7}$$

for real $r \geq 0$. Depending on the context we want these bounds $c_{W,v}$ to be either just positive or bounded away from 0 uniformly for all possible v and W and r in some interval. This encompasses the search for sufficient conditions on r to guarantee uniform bounds. In these cases we express the $c_{v,W}(r)$ explicitly. This allows further analysis in some interesting cases.

1.2 Vue d'ensemble

Cette thèse traite des propriétés et applications de certains champs de Bernoulli aléatoires, soumis à des conditions d'indépendance encodées par un graphe. Un *champ de Bernoulli aléatoire* (bref CBA) est une collection de variables aléatoires Bernoulli $Y := (Y_v)_{v\in V}$ indexée par les sommets d'un graphe $G := (V,E)$. L'exemple prototypique d'une telle condition est *l'indépendance forte relative à* G. C'est-à-dire les sous-champs non-adjacents de Y sont indépendants :

$$\forall\, U,W \subsetneq V : \quad d(U,W) > 1 \;\Rightarrow\; Y_U := (Y_v)_{v\in U} \text{ est indépendant de } Y_W\,.$$

Dans ce contexte G s'appelle aussi le *graphe de dépendance forte* de Y [LSS97, SS05, AS08]. On se rappelle, qu'un sous-ensemble $W \subseteq V$ est *indépendant* ssi il ne contient aucun paire des voisins. C'est-à-dire le sous-graphe $G(W)$ engendré par un ensemble indépendant ne contient que des sommets isolés. Un outil classique pour étudier les ensembles indépendants d'un graphe G fini est le *polynôme des ensembles indépendants* [HL94] :

$$I_G : \quad \mathbb{C} \to \mathbb{C} \quad z \mapsto \sum_{\text{indépendant } W\subseteq V} z^{|W|}\,.$$

C'est la *fonction generatrice* des ensembles indépendants de G. Nous nous restreignons pour la suite de cette section au contexte fini et homogène. Il y a deux CBAs particulièrement reliés à I_G.

Le premier CBA est connecté avec le *Lemme Local de Lovász* [EL75], un outil important de la *méthode probabiliste* en théorie des graphes [AS08]. Si $E \neq \emptyset$, alors la racine d'I_G la plus proche de 0 est $p_{sh}^G - 1 \in [-1,0]$ [SS05]. Pour $p \in [p_{sh}^G, 1]$ et $q := 1-p$ on définit la *mesure de Shearer* [She85] sur $\mathcal{X}_V := \{0,1\}^V$ par

$$\mu_{G,p}(Z = \vec{x}) := \begin{cases} q^{|W|} I_{G(\widehat{W})}(-q) & \text{si } W := V \setminus \operatorname{supp}(\vec{x}) \text{ est indépendant,} \\ 0 & \text{sinon,} \end{cases}$$

où $\widehat{W} := W \uplus \mathcal{N}(W)$ et $\mathcal{N}(W)$ sont les voisins de W. Notez que $\mu_{G,p}(Z = \vec{1}) = I_G(-q)$ et qu'elle a G comme graphe d'indépendance forte. Soit $\mathcal{C}_G^{\text{strong}}(p)$ la classe de tous les CBAs avec graphe de dépendance forte G et paramètre marginal p. L'importance de la mesure de Shearer vient du fait que pour tout $p > p_{sh}^G$, tout $Y \in \mathcal{C}_G^{\text{strong}}(p)$ et tout $U \subseteq W \subseteq V$ on a

$$\mathbb{P}(Y_W = \vec{1} | Y_U = \vec{1}) \geq \mu_{G,p}(Z_W = \vec{1} | Z_U = \vec{1}) = \frac{I_{G(W)}(-q)}{I_{G(U)}(-q)} > 0\,. \tag{1.8}$$

C'est-à-dire que toutes les probabilitiés conditionnelles ci-dessus sont bien définies et elles sont minorées par celles de la mesure de Shearer. Nous appliquons cette minimalité pour obtenir des resultats sur la percolation et la domination stochastique des champs de Bernoulli produits par des CBAs. À cette fin nous étudions la structure, le comportement et la représentation de la mesure de Shearer en détail.

Le second CBA est bien connu dans la mécanique statistique : c'est le modèle appelé *modèle du gaz de reseau à noyau dur*, ou aussi *modèle des animaux* [Dob96b] ou *modèle abstrait de polymères* [KP86, FP07]. Soit $z \in [0, \infty[$ la *fugacité* ou *le potentiel chimique*. Le *support* d'un vecteur $\vec{x} := (x_v)_{v \in V} \in \mathcal{X}_V$ est $\mathrm{supp}(\vec{x}) := \{v \in V : x_v = 1\}$. On définit la mesure de probabilité $\lambda_{V,z}$ de ce modèle sur $\mathcal{X}_V$ par

$$\lambda_{V,z}(\vec{x}) := \begin{cases} \frac{z^{|W|}}{I_G(z)} & \text{si } W := \mathrm{supp}(\vec{x}) \text{ est indépendant,} \\ 0 & \text{sinon.} \end{cases}$$

Dans ce cadre le facteur de normalisation I_G s'appele *fonction de répartition* du modèle. Ce modèle n'a pas G comme graphe de dépendance forte. Une statistique clé du modèle est *l'énergie libre* $-\log(I_G(z))/|V|$ [Rue69, Bax89]. Les méthodes de *développement en amas* [KP86, Dob96b, MS00] demandent aussi le contrôle des *corrélations avec accrochage* $I_{G(W)}(z)/I_G(z)$ pour $W \subseteq V$ et z complexe. Ainsi on veut un contrôle uniforme de la quantité en question dans un domaine complexe petit autour de 0 quand on prend la *limite thermodynamique*, c'est-à-dire quand $|V| \to \infty$. Nous développons des techniques d'opérateurs sur les arbres [FP07] pour donner des conditions suffisantes plus faibles pour un tel contrôle uniforme et nous discutons alors la relation avec l'analyticité de l'énergie libre en détail.

Vus à un niveau plus technique, les termes suivants jouent un rôle clé dans le traitement des sujets ci-dessus :

$$\forall\, v \notin W \subseteq V : \quad \frac{|I_{G(W)}(z)|}{|I_{G(W \uplus \{v\})}(z)|} \,.$$

On contrôle ces quantités en trouvant des bornes inférieures de la forme

$$\forall\, v \notin \text{fini } W \subseteq V : \quad \frac{|I_{G(W)}(-r)|}{|I_{G(W \uplus \{v\})}(-r)|} \geq c_{v,W}(r) \,,$$

pour $r \geq 0$ réel. Suivant le contexte, nous voulons que ces bornes soient ou bien positives, ou bien éloignées de 0 uniformément en v, W et r dans certains intervalles. Ceci impose la recherche de conditions suffisantes sur r pour garantir des bornes uniformes. Dans ces cas là nous exprimons les $c_{v,W}(r)$ explicitement. Ces expressions permettont une analyse plus profonde dans certains cas intéressants.

1.3 Übersicht

Diese Dissertation behandelt Eigenschaften und Anwendungen von Bernoulli-Zufallsfeldern, welche durch einen Graph gegebenen Unabhängigkeitsbedingungen unterworfen sind. Ein *Bernoulli-Zufallsfeld* (kurz BZF) ist eine Sammlung von Bernoulli Zufallsvariablen $Y := (Y_v)_{v \in V}$, indiziert durch die Knoten eines Graphen $G := (V, E)$. Das prototypische Beispiel einer solchen Unabhängigkeitsbedingung ist *starke Unabhängigkeit bezüglich* G, dass heisst nicht benachbarte Teilfelder von Y sind unabhängig:

$$\forall\, U, W \subsetneq V : \quad d(U, W) > 1 \;\Rightarrow\; Y_U := (Y_v)_{v \in U} \text{ ist unabhängig von } Y_W \,.$$

In diesem Kontext wird G auch *starker Abhängigkeitsgraph* von Y [LSS97, SS05, AS08] genannt. Eine Untermenge der Knoten $W \subseteq V$ ist genau dann *unabhängig*, wenn sie keine benachbarten Knoten enthält. Dass heisst, dass der von einer unabhängigen Menge W *induzierte Untergraph* $G(W)$ eine Ansammlung *isolierter Knoten* ist. Wenn Y den Graph G als starken Abhängigkeitsgraphen hat, dann indizieren die unabhängigen Mengen von G unabhängige Teilfelder von Y. Ein klassisches Werkzeug zur Untersuchung unabhängiger Mengen eines endlichen Graphen G ist das *Unabhängige-Mengen-Polynom* [HL94]:

$$I_G: \quad \mathbb{C} \to \mathbb{C} \quad z \mapsto \sum_{\text{unabhängig } W \subseteq V} z^{|W|}\,.$$

Es ist die *erzeugende Funktion* der unabhängigen Mengen von G. Für den Rest dieser Sektion verbleiben wir im endlichen und homogenen Kontext. Es gibt zwei BZF, welche besonders mit I_G verbunden sind.

Das erste BZF ist mit dem *Lovász Local Lemma* [EL75], einem wichtigen Werkzeug der *probalistischen Methode* in der Graphentheorie [AS08], verbunden. Wenn $E \neq \emptyset$, dann ist die am nähesten bei 0 liegende Wurzel von I_G gleich $p_{sh}^G - 1 \in [-1, 0]$ [SS05]. Für $p \in [p_{sh}^G, 1]$ und $q := 1 - p$ definiere *Shearer's Maß* [She85] auf $\mathcal{X}_V := \{0,1\}^V$ durch

$$\mu_{G,p}(Z = \vec{x}) := \begin{cases} q^{|W|} I_{G(\widehat{W})}(-q) & \text{falls } W := V \setminus \operatorname{supp}(\vec{x}) \text{ unabhängig ist,} \\ 0 & \text{sonst,} \end{cases}$$

wobei $\widehat{W} := W \uplus \mathcal{N}(W)$ und $\mathcal{N}(W)$ die Nachbarn von W sind. Insbesondere ist $\mu_{G,p}(Z = \vec{1}) = I_G(-q)$ und Shearer's Maß hat G als starken Abhängigkeitsgraphen. Sei $\mathcal{C}_G^{\text{strong}}(p)$ die Klasse aller BZF mit starkem Abhängigkeitsgraphen G und marginalem Parameter p. Die Wichtigkeit von Shearer's Maß liegt [She85] darin, dass für jedes $p > p_{sh}^G$, jedes $Y \in \mathcal{C}_G^{\text{strong}}(p)$ und alle $U \subseteq W \subseteq V$ gilt:

$$\mathbb{P}(Y_W = \vec{1} | Y_U = \vec{1}) \geq \mu_{G,p}(Z_W = \vec{1} | Z_U = \vec{1}) = \frac{I_{G(W)}(-q)}{I_{G(U)}(-q)} > 0\,.$$

Das heisst, dass alle obigen konditionellen Wahrscheinlichkeiten wohldefiniert sind und gleichmäßig durch jene von Shearer's Maß minimiert werden. Diese Minimalität erlaubt es, Fragestellungen für große Klassen von BZF auf Shearer's Maß zu reduzieren. Wir wenden obige Minimalität an, um Resultate über Perkolation und die stochastische Domination von Bernoulli-Produktfeldern durch BZF herzuleiten. Mit diesem Ziel vor Augen untersuchen wir die Struktur, das Verhalten und die Repräsentation von Shearer's Maß genau.

Das zweite BZF ist in der Statistischen Mechanik wohlbekannt: es ist das sogenannte Modell eines *harten Gittergases*, auch unter dem Namen *Tiermodell* [Dob96b] oder *abstraktes Polymermodell* [KP86, FP07] bekannt. Sei $z \in [0, \infty[$ die *Fugazität* oder das *chemische Potential*. Der *Support* eines Vektors $\vec{x} := (x_v)_{v \in V} \in \mathcal{X}_V$ ist $\operatorname{supp}(\vec{x}) := \{v \in V : x_v = 1\}$. Wir definieren das Wahrscheinlichkeitsmaß $\lambda_{V,z}$ dieses Modells auf $\mathcal{X}_V$ durch

$$\lambda_{V,z}(\vec{x}) := \begin{cases} \frac{z^{|W|}}{I_G(z)} & \text{falls } W := \operatorname{supp}(\vec{x}) \text{ unabhängig ist,} \\ 0 & \text{sonst.} \end{cases}$$

In diesem Kontext wird der Normalisierungsfaktor I_G *kanonische Zustandssumme* des Modells genannt. Dieses Modell hat jedoch nicht G als starken Abhängigkeitsgraphen. Eine Schlüsselgröße dieses Modells ist die *freie Energie* $-\log(I_G(z))/|V|$ [Rue69, Bax89]. *Cluster-Entwicklungstechniken* [KP86, Dob96b, MS00] verlangen gleichfalls nach einer Kontrolle der sogenannten *verankerten Korrelationen* $I_{G(W)}(z)/I_G(z)$ für $W \subseteq V$ und komplexes z. Aus diesen Gründen will man eine gleichmäßige Kontrolle der untersuchten Größe in einer kleinen komplexen Domaine um 0 beim Übergang in das *thermodynamische Limit*, dass heisst wenn $|V| \to \infty$. Wir verallgemeinern Baum-Operator Methoden [FP07] um schwächere hinreichende Bedingungen für eine solche gleichmäßige Kontrolle herzuleiten und diskutieren den Zusammenhang mit der Analyzität der freien Energie ausführlich.

Auf einem technischeren Niveau gesehen spielen die folgenden Verhältnisse eine Schlüsselrole in der Behandlungen der obigen zwei Thematiken:

$$\forall\, v \notin W \subseteq V : \quad \frac{|I_{G(W)}(z)|}{|I_{G(W \uplus \{v\})}(z)|}\,.$$

Die Kontrolle dieser Quantitäten wird durch das Auffinden unterer Grenzen der Form

$$\forall\, v \notin \text{finite } W \subseteq V : \quad \frac{|I_{G(W)}(-r)|}{|I_{G(W \uplus \{v\})}(-r)|} \geq c_{v,W}(r)\,,$$

für reelles $r \geq 0$, erreicht. Je nach Kontext wollen wir diese Grenzen $c_{W,v}$ entweder nur positiv oder gleichmäßig positiv für alle möglichen v und W und r in gewissen Intervallen. Dies inkludiert die Suche nach hinreichenden Bedingungen an r für solche gleichmäßigen Grenzen. In diesen Fällen drücken wir die $c_{v,W}(r)$ explizit aus. Dies erlaubt eine tiefergehende Analyse in einigen besonders interessanten Fällen.

1.4 Notation and conventions

We recall conventions and definitions throughout this thesis as needed. The common denominator though is:

Graphs are always undirected and locally finite. Given a subset A of the vertices and/or edges of a graph G we denote by $G(A) := (V(A), E(A))$ the *subgraph induced* by A. The neighbours of a vertex v and a subset of vertices W are $\mathcal{N}(v)$ and $\mathcal{N}(W)$ respectively.

Vectors are denoted by $\vec{x}$ and are indexed by a countable base set V. We apply *scalar operations component-wise* to vectors (as in $\vec{x}\vec{y} := (x_v y_v)_{v\in V}$ or $|\vec{x}| := (|x_v|)_{v\in V}$). For a subset W of the indices write $\vec{x}_W := (x_v)_{v\in W}$ for the *projection* of $\vec{x}$ onto the subvector indexed by W. We omit the projection if there is no ambiguity. For a constant c we let constant vector $\vec{c}$ be $c\,\vec{1}$. If we supply a scalar instead of a vector in a function expecting a vector we implicitly lift it to the appropriate constant vector. This is the *homogeneous setting*. We look at BRFs as vector-valued rvs and all conventions for vectors apply to them.

The notation in the literature around the LLL, in statistical mechanics and proability theory of Bernoulli random fields is varied and inhomogeneous. In the setting of Shearer's measure we exclusively talk about BRFs with marginal parameter vector $\vec{p}$. We always assume that $\vec{q} := \vec{1} - \vec{p}$, also in the homogenous setting and with common sub-/superscripts. In the statistical mechanics setting we restrict ourselves to the language of abstract polymer systems (see chapter 5), not to be confused with the polymers in section 1.6.3.

We take the liberty of mixing BRFs and their laws, in particular when talking about stochastic domination. We also take the shortcut (as in (1.4)) to define $\mu_{G,p}$ on the canonical probability space $Z := (Z_v)_{v\in V}$ and to refer at the same time to Z as a $\mu_{G,p}$-distributed BRF, switching notions as more convenient.

Instead of indicator functions we extensively use *Iverson brackets*.

We express the fact that A is a *finite subset* of a set B by $A \Subset B$. Likewise we write $H \ll G$ for H being a finite subgraph of G. When we take limits of such finite subsets and -graphs *exhausting* their infinite superset and -graph we write $A_n \nearrow B$ and $H_n \nearrow G$ respectively. If necessary, we understand the limit of exhausting subgraphs to be in the sense of *van Hove*, that is along a *Følner sequence* $(H_n)_{n\in\mathbb{N}}$ with

$$\lim_{n\to\infty} \frac{|\partial H_n|}{|H_n|} = 0\,.$$

We often deal with the space R^V, where R is a real interval (maybe extended with $\pm\infty$) or subset of $\mathbb{C}$ and V is countable. The topology on this space is always the *box topology*, that is

$$\vec{x}^{(n)} \to \vec{x} \quad \Leftrightarrow \quad \forall\, v \in V : x_v^{(n)} \to x_v\,. \tag{1.9}$$

1.5 The Lovász Local Lemma, Shearer's measure and applications

1.5.1 The Lovász Local Lemma

The aim of the probabilistic method [AS08] is to show the *existence* of a combinatorial object with certain properties. This is done by constructing an appropriate probability space and showing that such an object exists with nonzero probability. Usually, as in *extremal graph theory*, the object studied is finite and hence the probability space isomorph to a finite-dimensional vector space.

Suppose that the desired properties of the object are implied by a collection of *local properties* of the object. Local in the sense that a local property only depends on a small substructure of the object. The satisfiability of local properties depending on disjoint substructures are thus independent. Examples of such local properties are: the determinant of a 2×2 minor of a matrix M (non-overlapping minors do not use the same entries of M) or left-to-right crossings in small neighbourhoods of a finite subgraph of $\mathbb{Z}^2$ (non-overlapping neighbourhoods are independent) or the existence of Latin transversals of a square matrix [BFPS11]. The classical example is 5 vertices of a graph forming a *clique*, that is inducing K_5 as a subgraph. This is connected to bounds on *Ramsey numbers* [Ram30, AS08].

We formalize this in the random setting as follows: Let $G := (V, E)$ be a finite simple graph and $Y := (Y_v)_{v \in V}$ be a collection of $\{0,1\}$-valued rvs. For each v the event $\{Y_v = 1\}$ corresponds to the fact the local property v holds. The *dependency graph* G encodes the independence of the local properties: we assume for two disjoint subsets U, W of V that Y_U is independent of Y_W iff $d(W, U) > 1$, that is there are no edges between W and U: $E(W, U) = \emptyset$. Thus $Y \in \mathcal{C}_G^{\text{strong}}(\vec{p})$, where $p_v := \mathbb{P}(Y_v = 1)$. We want to

Problem 1. Find conditions on $\{Y_v\}_{v \in V}$ to imply that $\mathbb{P}(Y_V = \vec{1}) > 0$.

Rare dependencies between the $\{Y_v\} v \in V$ correspond to a dependency graph G with few edges. For such cases Erdös and Lovász [EL75] have derived a sufficient condition on $\vec{p}$ to guarantee that $\mathbb{P}(Y_V = \vec{1}) > 0$.

The Lovász Local Lemma (short LLL). *If there exists $\vec{x} \in [0,1]^V$ such that*

$$\forall\, v \in V: \quad q_v \leq (1 - x_v) \prod_{w \in \mathcal{N}(v)} x_w\,, \tag{1.10}$$

then

$$\forall \text{ finite } U \subseteq W \subseteq V: \quad \mathbb{P}(Y_W = \vec{1} | Y_U = \vec{1}) \geq \prod_{v \in W \setminus U} x_v\,. \tag{1.11}$$

Note that the LLL depends only on the marginal parameter $\vec{p}$ and the dependency structure imposed by G. An immediate corollary to the LLL for dependency graphs with uniformly bounded degree D is the following sufficient condition on $\vec{p}$:

$$\forall\, v \in V: \quad q_v \leq \frac{1}{e(D+1)} \leq \frac{D^D}{(D+1)^{(D+1)}}\,. \tag{1.12}$$

This is derived from the LLL by setting $\vec{x} := \frac{1}{D+1}\vec{1}$.

In the literature on the LLL the focus is often on *not fulfiling* a number of local properties. By symmetry this leads to a dual version of Y with swapped marginal parameters $\vec{1} - \vec{p}$. The LLL, as the probabilistic method in general, is highly non-constructive. A line of research on the interface with computer science is to derandomize these existence results to obtain algorithms [AS08, section 5.7 & section 16],[Bec91],[Alo91],[Sri08], [MT10] to construct concrete instances with the desired properties.

There is a series of extensions and variations to the LLL. They include

- *lop-sided* versions, which relax the dependency conditions [AS08, page 70]. This is the case of a weak dependency graph [LSS97, SS05], or see section 2.1.1.
- *directed versions* [AS08, lemma 5.11], that is independence is only guaranteed from neighbours with incoming edges, for example.

- *compound versions* with additional conditions [Sri08], in particular local conditions seens as aggregates of other local conditions. This is like a two-stage version of problem 1.

1.5.2 Shearer's measure

Investigating the class $\mathcal{C}_G^{\text{strong}}$ of BRFs with strong dependency graph G in more detail, Shearer [She85] discovered that

Summary 2. *There exists an open, non-empty connected subset $\mathcal{P}_{sh}^G$ of $[0,1]^V$, such that the following dichotomy holds:*

$$\forall\,\vec{p} \in \mathcal{P}_{sh}^G : \exists!\, Z \in \mathcal{C}_G^{strong}(\vec{p}) : \forall\, Y \in \mathcal{C}_G^{strong}(\vec{p}), U \subseteq W \subseteq V :$$
$$\mathbb{P}(Y_W = \vec{1} | Y_U = \vec{1}) \geq \mathbb{P}(Z_W = \vec{1} | Z_U = \vec{1}) > 0 \tag{1.13a}$$

and

$$\forall\,\vec{p} \notin \mathcal{P}_{sh}^G : \exists\, Y \in \mathcal{C}_G^{strong}(\vec{p}) : \quad \mathbb{P}(Y_V = \vec{1}) = 0\,. \tag{1.13b}$$

We call the law of Z in (1.13a) *Shearer's measure* and denote it by $\mu_{G,\vec{p}}$. The minimality (1.13a) reduces problem 1 to

Problem 3. What are sufficient conditions for $\vec{p}$ to lie in $\mathcal{P}_{sh}^G$? What are the properties of $\mathcal{P}_{sh}^G$?

The LLL is one such condition, because if $\vec{p}$ fulfils (1.10), then $\vec{p} \in \mathcal{P}_{sh}^G$. Not only did Shearer derive (1.13), but he used $\mu_{G,\vec{p}}$ to show that the LLL is *asymptotically optimal* [She85, theorem 2]. In the homogenous case identify $\mathcal{P}_{sh}^G$ with the interval $]p_{sh}^G, 1]$. Then

Summary 4. *For every exhausting sequence $(\mathbb{T}_n)_{n\in\mathbb{N}}$ of finite subtrees of the D-regular infinite tree $\mathbb{T}_D$, we have*

$$\lim_{n\to\infty} p_{sh}^{\mathbb{T}_n} = 1 - \frac{D^D}{(D+1)^{(D+1)}}\,.$$

This matches the sufficient condition (1.12) and shows its asymptotic optimality. It is possible to *characterize* the law of Z in (1.13a), that is $\mu_{G,\vec{p}}$, succinctly:

Summary 5. *The law $\mu_{G,\vec{p}}$ is characterized by the two properties:*

- $\mu_{G,\vec{p}} \in \mathcal{C}_G^{strong}(\vec{p})$.
- *All realizations with adjacent* 0*s have zero mass under* $\mu_{G,\vec{p}}$.

This characterization implies the construction in (1.3), generalized in the obvious way for inhomogeneous $\vec{p}$. We give a detailed introduction to Shearer's measure in chapter 2.

1.5.3 Application to k-dependent percolation on trees

The first application and my initial motivation for my foray into this topic has been work on k-independent percolation on trees, started during my master thesis [Tem08] and culminating in [MT12]. This work is in the tradition of [Lyo90] and extends [BB06]. I just quote the abstract of [MT12], which forms chapter 3:

> Consider the class of k-independent percolations with parameter p on an infinite tree $\mathbb{T}$. We derive tight bounds on p for both a.s. percolation and a.s. nonpercolation. The bounds are continuous functions of k and the branching number $br(\mathbb{T})$ of $\mathbb{T}$. This extends previous results by Lyons for the independent case ($k = 0$) and by Balister & Bollobás for 1-independent bond percolations. Central to our argumentation are moment method bounds á la Lyons supplemented by explicit percolation models á la Balister & Bollobás. An indispensable tool is the minimality and explicit construction of Shearer's measure on the k-fuzz of $\mathbb{Z}$.

1.5.4 Application to domination of Bernoulli product fields

We call a Bernoulli product field (short BPF) with marginal parameter vector $\vec{p}$ *non-trivial* iff $\vec{p} > \vec{0}$. A stricter version of problem 1 is to ask [LSS97]:

Problem 6. For which $\vec{p}$ does every BRF Y with dependency graph G dominate stochastically a non-trivial BPF?

An even stronger version is

Problem 7. For which $\vec{p}$ does every BRF Y with dependency graph G dominate stochastically a non-trivial BPF with the same parameter?

Here the notion of dependency graph is weakened to allow even more BRFs. It turns out that the solution to these and related problems is $\mathcal{P}_{sh}^{G}$. We state here just the abstract of [Tem12], which forms chapter 4:

> Let $G := (V, E)$ be a locally finite graph. Let $\vec{p} \in [0, 1]^V$. We show that Shearer's measure, introduced in the context of the Lovász Local Lemma, with marginal distribution determined by $\vec{p}$ exists on G iff every Bernoulli random field with the same marginals and dependency graph G dominates stochastically a non-trivial Bernoulli product field. Additionaly we derive a non-trivial uniform lower bound for the parameter vector of the dominated Bernoulli product field. This generalizes previous results by Liggett, Schonmann & Stacey in the homogeneous case, in particular on the k-fuzz of $\mathbb{Z}$. Using the connection between Shearer's measure and hardcore gases established by Scott & Sokal, we apply bounds derived from cluster expansions of lattice gas partition functions to the stochastic domination problem.

1.6 The hard-core model in statistical mechanics

We give a short introduction to the language of statistical mechanics in section 1.6.1. The definition and discussion of the hard-core lattice gas follow in section 1.6.2. We then present two different cluster expansions related to the hard-core lattice gas. Section 1.6.3 shows the expansion of the partition function of the lattice gas itself, whereas section 1.6.4 reduces other models to the lattice gas, by building up an appropriate structure on abstract polymers.

1.6.1 The language of statistical mechanics

Statistical mechanics aims to reproduce the qualitative and quantitative behaviour of physical systems consisting of many particles or nearly independent subsystems in a formal and rigorous setting. The aim is to relate the local microscopic mechanics of the system (like interaction between two atoms) to the statistical macroscopic observable properties of the system (like pressure or entropy).

For a more comprehensive introduction see [Bax89, chapter 1] or [Rue69, chapter 1]. Here I only want to review the basic terminology for discrete systems in *thermadynamic equilibrium*, that is the system exchanges no energy with its surrounding and can be regarded as essentially isolated. Our system consists of a *finite volume* V of locations. Each location can be in a *state* taken from a countable set E. Thus all possible *configurations* of our model are elements of $\Omega := E^V$. We assign to each configuration $\omega \in \Omega$ an *energy* $H(\omega)$, called the *Hamiltonian* of ω. We are also given the absolute temperature T in kelvin and the *Boltzman constant* k. The standard formulation, introduced in the late 19^{th} century by the american physicist and chemist Josiah Willard Gibbs, is the *Boltzmann distribution* on Ω:

$$\mu_V(\omega) := Z_V^{-1}(T)\exp(-H(\omega)/kT)d\omega\,, \tag{1.14a}$$

where the normalizing factor Z_V is the so-called *partition function*, defined by

$$Z_V(T) := \sum_{\omega\in\Omega}\exp(-H(\omega)/kT)d\omega\,. \tag{1.14b}$$

This is sometimes called a *perturbation* of the à priori measure $d\omega$ on Ω. If E is finite then $d\omega$ is usually just the uniform counting measure. Note that as $T \to 0$ the configurations with minimal energy gain mass under μ_V, corresponding to a *frozen* phase, while if $T \to \infty$ the differences in the hamiltonian matter less and the mass of μ_V is spread more uniformly over Ω, corresponding to a *plasma* phase. A key quantity in the analysis of the system is the *free energy*

$$F_V(T) := -\frac{\log Z_V}{kT}\,. \tag{1.15}$$

Its and its derivates' behaviour determine the macroscopic properties of the system [Sim93, Rue69].

1.6.2 The hard-core lattice gas

One of the basic models in the setting of section 1.6.1 is the so-called *hard-core lattice gas* [Bax89, Rue69]. It also called the *animal model* [Dob96b] or *abstract polymer model* [KP86, FP07]. It describes the possible configurations of particles (atoms of the gas) on a the vertices (the *locations*) of a finite graph $G := (V, E)$. The classic case is G being a finite subgraph of a regular d-dimensional lattice. The configurations are subject to two restrictions: each location can carry at most one particle (*self-exclusion* or *ard-core self-repulsion*) and neighbouring locations can not carry particles (*hard-core interaction*). Thus the locations of particles in an admissible configuration corresponds to an independent set of G. Additionaly a vector of *fugacities* (or *chemical potentials*) $\vec{z} \in \mathbb{R}_+^V$ is given, describing the a priori density of particles at the locations. The probability measure is given by

$$\lambda_{G,\vec{z}}(A) := \begin{cases} \frac{\prod_{v\in A} z_v}{\Xi_G(\vec{z})} & \text{if } A \text{ is independent} \\ 0 & \text{else,} \end{cases} \tag{1.16a}$$

with partition function

$$\Xi_G(\vec{z}) := \sum_{A \text{ independent}} \prod_{v\in A} z_v \,. \tag{1.16b}$$

A derivation of (1.16a) in the setting of 1.6.1 via *pair potentials* is given in [SS05, section 1.1]. The marginal parameters of $\lambda_{G,\vec{z}}$ are

$$\lambda_{G,\vec{z}}([v \in A]) = \frac{z_v \, \Xi_{G(V \setminus \mathcal{N}(v))}(\vec{z})}{\Xi_G(\vec{z})} \tag{1.16c}$$

Thus the marginal parameters are related to (1.6).

The measure $\lambda_{G,\vec{z}}$ has a *spatial Markov property*. Partition V into three sets A, B and C with $d(A, B) > 1$, that is C separates A and B. Then

$$\forall\, \vec{s} \in \mathcal{X}_V : \quad \lambda_{G,\vec{z}}(\vec{s}_A, \vec{s}_B | \vec{s}_C) = \lambda_{G,\vec{z}}(\vec{s}_A | \vec{s}_C) \lambda_{G,\vec{z}}(\vec{s}_B | \vec{s}_C) \,. \tag{1.17}$$

Therefore it is a *Markov random field*. It also fulfils the Dobrushin,Langford & Ruelle conditions [Geo11, section 2], which imply (1.17) on finite subgraphs. The DLR conditions imply the existence of one or more so-called *infinite volume Gibbs measures* having $\lambda_{G,\vec{z}}$ as its finite projection for every finite subgraph G.

The classical questions are again centered around the behaviour of $\Xi_G(\vec{z})$, the free energy $F_G(\vec{z}) := -\frac{1}{|V|} \log \Xi_G(\vec{z})$ and the *correlation* of the states at $W \subseteq V$, given by $\frac{\Xi_{G(V\setminus\{W \uplus \mathcal{N}(W)\})}(\vec{z})}{\Xi_G(\vec{z})}$.

1.6.3 Cluster expansions - convergence of the lattice gas

The *high-temperature expansion* of the model is an expansion of $F_G(\vec{z})$ around $\vec{z} = \vec{0}$ in complex variables. The main focus (see also section 1.6.4) is on the analyticity of F_G in a small complex multi-disc around $\vec{0}$, uniform in the system size.

In the particular case of some homogeneous and regular 2-dimensional lattices a so-called *non-physical singularity* [Woo85, Gut87, Tod99, Yb11],[SS05, section 8] limits this multi-disc (here just a disc) for negative real fugacities. It is my understanding that the mathematical physics community considers this singularity to be related to the limit of the zeros of the finite size partition functions, but no formal proof of this has come to my attentation yet. We rectify this situation in section 5.2.3.

One technique to derive bounds as in 1.7 is to expand the clusters in the partition function Ξ_V of the hardcore gas. The key step is to rewrite the partition function as a *weighted exponential generating function*:

$$\Xi_V(\vec{z}) = \sum_{n\geq 0} \frac{1}{n!} \sum_{\vec{v}\in V^n} \left(\prod_{1\leq i<j\leq n} [v_i \not\sim v_j] \right) \prod_{i=1}^{n} z_{v_i} \,. \tag{1.18}$$

This can be seen as a sum over subgraphs of V. Taking the logarithm amounts to summing only over connected subgraphs, as every graph can be uniquely decomposed into its conneced components (see [Rue69, section 4.4],[MS10, Far10]). This is the so-called *Mayer expansion* [MM41][SS05, section 2.2] of the mathematical physics community.

A detailed investigation of the terms

$$\sum_{\vec{v}\in V^n} \left(\prod_{1\leq i<j\leq n} [v_i \not\sim v_j] \right) \prod_{i=1}^{n} z_{v_i} \,, \tag{1.19}$$

called the *truncated Ursell functions*. There is a long history [KP86, Cam82, Dob96a, FP07, MS10] of studying bounds on them to get improved bounds of the form (1.7). Chapter 5 presents such an example.

1.6.4 Cluster expansion - reduction to the lattice gas

Given a model with short range interactions a popular way to investigate its partition function is to apply the *cluster expansion* [Far10, section 6.2],[Dob96b, sections 1.3-1.5]. Examples are *perturbations* of iid models. At its core is the so-called *cluster representation* of the partition function in terms of certain geometrical objects, called *polymers* (or *contours* in dimension 2). Let $G := (V, E)$ be finite. Then

$$Z_G(\vec{z}) = \sum_{\gamma\in\Gamma} \prod_{A\in\gamma} w_A(\vec{z}) \,. \tag{1.20}$$

Here γ are the polymers (that is hypergraphs on subsets of V) and $w_A(\vec{z})$ are the cluster coefficients (subsets of V being the elements of γ). Here $w_A(\vec{z})$ *may be negative real*, too. A polymer γ may consist of several clusters, which are subsets of V. If we look at the free energy we get

$$\frac{1}{|V|} \log Z_G(\vec{z}) = \sum_{\gamma\in\Gamma} \prod_{A\in\gamma} w_A(\vec{z}) \prod_{B\in\gamma} [A\cap B = \emptyset] \,. \tag{1.21}$$

The logarithm can be understood as decomposing γ into its connected components. A graph structure on the clusters is given by $A \not\leftrightarrow B$ iff $A\cap B = \emptyset$. In this case A and B are said to be *compatible*. This gives rise to a graph $\mathcal{P}(G)$. Thus we have

$$\log Z_G(\vec{z}) = \Xi_{\mathcal{P}(G)}(\vec{w}(\vec{z}))\,, \tag{1.22}$$

where the rhs partition function is for a hardcore gas on $\mathcal{P}(G)$. This rewriting motivates the need to control Ξ_G for the hardcore gas as $V(G)$ grows and in a complex domain around $\vec{0}$.

In chapter 5 we discuss the *abstract polymer model*, which is can be thought of as both a generic cluster representation or a hardcore gas. Take note, though, that the clusters in the cluster representation are the polymers in the abstract polymer model. This naming is inspired by the polymer model [Dob96b, section 3], which is in straightforward bijection with the hardcore gas. At this point I have to quote [Far10, page 196], who nailed the often confusing terminology with:

> The terminology is inherently confusing, since when the cluster representation is interpreted as a grand partition function, then it is possible to do a cluster expansion of the cluster representation. That is, the cluster representation represents combinatorial coefficients in terms of cluster coefficients associated with connected clusters of sites. Then this representation is reinterpeted as a grand partition function, which has a cluster expansion indexed by connected clusters of polymers.

1.6.5 The hard-sphere model

Let Λ be a *finite volume* of $\mathbb{R}^d$, that is a bounded and measurable subset of $\mathbb{R}^d$ with respect to Lebesgue measure. The *hard-sphere gas* of *diameter* $R \in]0,\infty[$ enclosed in Λ at *density* z [FPS07] has the grand canonical partition function

$$\mathcal{Z}_\Lambda(z): \quad z \mapsto \sum_{n\geq 0} \frac{z^n}{n!} \int_{\Lambda^n} \exp\left(- \sum_{1\leq i<j\leq n} U(x_i - x_j)\right) dx_1 \cdots dx_n\,, \tag{1.23a}$$

where the *pair potential function* U given by

$$U: \quad \mathbb{R}_+ \to \mathbb{R}_+ \quad r \mapsto \begin{cases} 0 & \text{if } r > R \\ \infty & \text{if } r \leq R\,. \end{cases} \tag{1.23b}$$

It is the potential taking on the value ∞ for small distances that qualifies the gas as "hard" by disallowing configurations with pairs of molecules at distance less than R. In two dimensions it is called the *hard disc* model. This is not a discrete system as in section 1.6.1 anymore. Configurations correspond to finite sets of points in Λ. As in the discrete case one wants to investigate

$$\lim_{\Lambda\nearrow\mathbb{R}^d} -\frac{1}{\mathrm{vol}(\Lambda)} \log \mathcal{Z}_\Lambda(z)\,. \tag{1.23c}$$

The configurations of the hard-sphere gas are finite sets of points of Λ. The hard-core repulsion encoded by U allows us to define a finite graph G_A induced by a configuration A:

$$G(A) := (A, \{(a,b) \in A^2 : |a-b| \leq R\})\,. \tag{1.23d}$$

This way we can write

$$\exp\left(-\sum_{1\leq i<j\leq n} U(x_i - x_j)\right) = \prod_{1\leq i<j\leq n} [v_i \not\sim v_j]\,. \tag{1.23e}$$

Remark that this is the same expression as in (1.18) partition function of the hard-core gas. The main difference is that we do not sum over a countable number of configurations but integrate over them. This should motivate the close connection between the hard-core gas and the hard-spheres model. One can transfer methods like the cluster expansion in section 1.6.3 and carry estimates for the Ursell functions from the discrete to the continuous case.

1.7 Summary and discussion of technical results

This section aims to give an overview of the technical results of this thesis. That is we leave aside the results from the application of Shearer's measure in chapters 3 and 4 and refer the interested reader to these chapters. Instead we focus on the part common to all parts of the thesis, that is results which are essentially statements about I_G. We omit all derivative results or collorarries and just focus on bounds of the form (1.7).

Let $G := (V, E)$ be a (possibly infinite) graph and let $\vec{r} \in [0, \infty[^V$. As indicated in (1.6) the focus is on the quantities

$$\forall\, v \notin W \Subset V : \quad \mathcal{A}^v_W(\vec{r}) := \frac{|I_{G(W \uplus \{v\})}(-\vec{r})|}{|I_{G(W)}(-\vec{r})|}\,. \tag{1.24}$$

It is known that they enjoy some nice *monotonicity properties*: they are monotone decreasing in both $\vec{r}$ and W. Denote the *set of admissible parameters* by

$$\begin{aligned}\mathcal{R}_G :=&\{\vec{r} \in [0, \infty[^V : \quad \forall\, H \ll G : \qquad I_H(-\vec{r}) > 0\} \\ =&\{\vec{r} \in [0, \infty[^V : \quad \forall\, v \notin W \Subset V : \quad \mathcal{A}^v_W(\vec{r}) > 0\}\,.\end{aligned} \tag{1.25}$$

Known properties of $\mathcal{R}_G$ are that it is a subset of $[0, 1]^V$, is *log-convex* and a *down-set*. It decreases if G increases. In the finite case it is open, but in the infinite case it can be written as the intersection of a countable of open sets is in general not open anymore.

The two main questions are

Question 8. What is the shape of $\mathcal{R}_G$? What are sufficient or necessary conditions for $\vec{r} \in \mathcal{R}_G$?

and

Question 9. What are sufficient or necessary conditions for $\vec{r}$ to bound $\mathcal{A}^v_W(\vec{r})$ uniformly away from 0?

We analyse this question along both quantitative and qualitative lines. Qualitative results include monotone behaviour (see lemma 117 and propositions 21, 24, 32, 26 and 178) and the worst case behaviour for negative real arguments

Name/Origin	If $r_v \leq$	Then $\mathcal{A}^v_W(\vec{r}) \geq$
Kotecký & Preiss [KP86]	$s_v \exp\left(-\sum_{w\in\mathcal{N}_1(v)} s_w\right)$	$(1-r_v)^{\exp(\sum_{w\in\mathcal{N}_1(v)} s_w)-1}$
classic LLL [EL75], Dobrushin [Dob96b]	$s_v \prod_{w\in\mathcal{N}_1(v)} (1+s_w)^{-1}$	$\frac{1}{1+s_v}$
symmetric LLL [AS08], Dobrushin [Dob96b]	$\frac{D^D}{(D+1)^{(D+1)}}$	$\frac{D}{D+1}$
tree-iterative [FP07]	$\frac{s_v}{I_{G(\mathcal{N}_1(v))}(\vec{s})}$	$(1-r_v)^{I_{G(\mathcal{N}_1(v))}(\vec{s})-1}$
[SS05] and [LSS97], inspired by Shearer [She85], only escaping (W,v)	$\frac{(D-1)^{(D-1)}}{D^D}$	$\frac{D-1}{D}$
new inductive, only escaping (W,v) with escape $w\in\mathcal{N}(v)\setminus W$	$\min_{w\in\mathcal{N}(v)} \frac{s_v}{\prod_{u\in\mathcal{N}_1(v)\setminus\{w\}}(1+s_u)}$	$\frac{1}{1+s_v}$
new generic, iff $\vec{s}\in\mathcal{R}_G$	s_v	$\frac{r_v}{s_v-r_v}$
new tree-iterative, only escaping (W,v) with escape $w\in\mathcal{N}(v)\setminus W$	$\min_{w\in\mathcal{N}(v)} \frac{s_v}{I_{G(\mathcal{N}_1(v)\setminus\{w\})}(\vec{s})}$	$(1-r_v)^{I_{G(\mathcal{N}_1(v)\setminus\{w\})}(\vec{s})-1}$

Table 1.1: Summary of some sufficient conditions for $\vec{r}$ to lie in $\mathcal{R}_G$. The condition may on $\vec{r}$ may demand the existence of a suitable $\vec{s}\in]0,\infty[^V$.

(see section 5.8.4). We also deduce a generic condition to be non-zero (see proposition 124 and compare with section 2.2.3). There is also a rigorous clarification of the relation with solution of the numeric search for negative singularities via *transfer-matrix* techniques in section 5.2.3. Finally the partition scheme in section 6.3.3 finds a lazy version of the pruned SAW tree-interpretation by Scott & Sokal on the polymer level in section 6.2.3 in a partition of the clusters. Quantitative results are given in the following section 1.7.1.

1.7.1 Summary of bounds

Let $\vec{r},\vec{s}\in[0,\infty[$. Let D be the maximal degree of a vertex in G. Here G may be infinite, too. The condition column is for every v and the implication column for all (W,v) with $v\in W$ and $W\subseteq V$ finite. We call such a pair (W,v) *escaping* iff $\mathcal{N}(v)\setminus W\neq\emptyset$.

There are also exact solutions for some particular graphs in the homogeneous case. The classic one is the D-regular tree (inspired by [She85]). If

$r \leq \frac{(D-1)^{(D-1)}}{D^D}$ and ξ be the unique solution of $(1-\xi)\xi^{D-1}$ with $\xi \geq \frac{D-1}{D}$, then $\mathcal{A}_W^v(r) \geq \xi$ for every escaping (W, v). Similarly we can treat the k-fuzz of $\mathbb{Z}$ (inspired by [LSS97]). If $r \leq \frac{k^k}{(k+1)^{(k+1)}}$ and ξ be the unique solution of $(1-\xi)\xi^k$ with $\xi \geq \frac{k}{k+1}$, then $\mathcal{A}_W^v(r) \geq \xi$ for every escaping (W, v).

The classic conditions try to exert control over all (W, v) at the same time. Newer conditions allow to ignore one of the neighbours of v, giving only control over escaping pairs. To still allow some, albeit far from optimal control, the new generic condition allows to take the parameters for control of the escaping (W, v), reduce those parameters a bit and then control all (W, v). The amount of control one gets depends on by how much reduces the parameters.

Calculations and evaluations of these bounds for transitive grid-like graphs, in particular $\mathbb{Z}^2$, are in section 6.5. Finally, a condition for $\vec{r} \notin \mathcal{R}_G$ is given in theorem 174. It allows us to show the asymptotic behaviour for the graph $\mathbb{Z}^d$, as $d \to \infty$, in corollary 193 in section 6.4.1.

1.7.2 A translation guide

This section contains a translation guide between the notations used in the different chapters. The left column is the notation for the generic setting in the introduction in chapter 1, the middle column is the polymer system notation in chapter 5 and parts of chapter 6 and the right column is the notation for Shearer's measure in chapters 2, 3, 4 and parts of chapter 6.

generic setting	polymer system	Shearer/BRF
$G := (V, E)$	$(\mathcal{P}, \approx)$	$G := (V, E)$
$\vec{r}$	$\vec{\rho}$	$-\vec{q} = \vec{p} - \vec{1}$
$\mathcal{R}_G$	$\mathcal{R}_{\mathcal{P}}$	$\mathring{\mathcal{Q}}_G^{sh} = \vec{1} - \mathring{\mathcal{P}}_{sh}^G$
$I_G(-\vec{r})$	$\Xi_\Lambda(-\vec{\rho})$	$\Xi_G(\vec{p})$
(W, v)	(Λ, γ)	(W, v)
$\mathcal{A}_W^v(\vec{r})$ with $v \notin W$	Φ_Λ^γ with $\gamma \in \Lambda$	$\alpha_W^v(\vec{r})$ with $v \notin W$.

Table 1.2: Translation guide between the notations for the generic setting in chapter 1, the polymer system of chapter 5 and Shearer's measure in chapter 2

Chapter 2

Shearer's measure

The first half of this chapter introduces Shearer's measure. We review its construction and basic properties on a finite graph and characterize it in section 2.2.1. The importance of Shearer's measure comes from its minimizing property in section 2.2.5 and the Lovász Local Lemma in section 2.2.6. We establish the existence and uniqueness of Shearer's measure on infinite graphs in section 2.3 and extend various results. We also review known and introduce new couplings involving Shearer's measure in section 2.2.3 and show that it is non-markovian in section 2.2.7. We introduce the one-vertex open extension probabilities (OVOEPs) in section 2.2.4 and employ them consistently as the core tool in our investigation (see sections 2.2.5, 2.6.1, 4.7 and 6.4.4).

The second half of this chapter deals with the question of how to determine the critical values and how to represent and construct Shearer's measure explicitly. Examples on particular finite graphs and state of the art sufficient conditions are given in section 2.5. We then turn to a class of quasi-transitive tree-like graphs, which are particularly amenable to analysis. Important members include the k-fuzz of $\mathbb{Z}$ and the d-regular tree.

Another important class of graphs, namely grid-like graphs, is investigated in chapter 6. One particular BPF dominated by Shearer's measure, the so-called *intrinsic domination*, is discussed in section 4.7.

2.1 Notation and conventions

Let $G := (V, E)$ be a locally finite graph. Denote by $\mathcal{N}(v)$ the *set of neighbours* of v and by $\mathcal{N}_1(v) := \mathcal{N}(v) \uplus \{v\}$ the *neighbourhood of v including v itself*. For every subset H of vertices and/or edges of G denote by $V(H)$, $E(H)$ and $G(H)$ the *vertices, edges and subgraph induced by H* respectively.

Vectors are indexed by V, i.e. $\vec{x} := (x_v)_{v \in V}$. Scalar operations on vectors act coordinate-wise (as in $\vec{x}\,\vec{y}$) and scalar comparisons hold for all corresponding coordinates of the affected vectors (as in $\vec{0} < \vec{x}$). For $W \subseteq V$ let $\vec{x}_W := (x_v)_{v \in W}$, where needed for disambiguation. We otherwise ignore superfluous coordinates. If we use a scalar x in place of a vector $\vec{x}$, then we mean to use $\vec{x} = x\vec{1}$ and call this the *homogeneous setting*. We *always assume the relation* $q = 1 - p$, also in vectorized form and when having corresponding subscripts. Denote by $\mathcal{X}_V := \{0,1\}^V$ the compact *space of binary configurations* indexed by V. Equip $\mathcal{X}_V$ with the natural partial order induced by $\vec{x} \leq \vec{y}$ (isomorph to the partial order induced by the subset relation in $\mathcal{P}(V)$).

A *Bernoulli random field* (short BRF) $Y := (Y_v)_{v\in V}$ on G is a rv taking values in $\mathcal{X}_V$, seen as a collection of Bernoulli rvs Y_v indexed by V. A *Bernoulli product field* (short BPF) X is a BRF where $(X_v)_{v\in V}$ is a collection of independent Bernoulli rvs. We write its law as $\Pi^V_{\vec{x}}$, where $x_v := \Pi^V_{\vec{x}}(X_v = 1)$.

A subset A of the *space of binary configurations* $\mathcal{X}_V$ or the space $[0,1]^V$ is an *up-set* iff

$$\forall\, \vec{x} \in A, \vec{y} \in \mathcal{X}_V : \quad \vec{x} \leq \vec{y} \Rightarrow \vec{y} \in A\,.$$

Replacing $\leq$ by $\geq$ yields a *down-set*.

If Y and Z are two independent BRFs, then we denote by

$$Y \wedge Z := (Y_v \wedge Z_v)_{v\in V} \qquad Y \vee Z := (Y_v \wedge Z_v)_{v\in V} \tag{2.1}$$

the *vertex-wise minimum* and *vertex-wise maximum* of Y and Z respectively.

2.1.1 Dependency graphs and classes

This section deals with the encoding of dependencies (or more correctly, independence) by a graph.

Definition 10. A random field $Y := \{Y_v\}_{v\in V}$ on a graph $G := (V, E)$ has a *strong dependency graph* G iff

$$\forall\, U, W \subset V : \quad d(U, W) > 1 \Rightarrow Y_U \perp Y_W\,, \tag{2.2a}$$

has *weak dependency graph* G iff

$$\forall\, v \in V : \quad Y_v \perp Y_{V\setminus\mathcal{N}_1(v)}\,, \tag{2.2b}$$

and is *pairwise independent* on G iff

$$\forall\, v, w \in V : \quad v \not\sim w \Rightarrow Y_v \perp Y_w. \tag{2.2c}$$

Having a strong dependency graph G is equivalent to being 1–independent on G. Adding edges to a dependency graph does not change its status as dependency graph, for a given random field Y. Example 15 shows, that in both the strong and weak case there may be several different minimal dependency graphs. The definitions (2.2) are listed in order of decreasing strength. Example 14 shows, that pairwise independence is not sufficient for weak independence. The comment after theorem 106 or [LSS97, page 89] shows, that there are BRFs with a dependency graph G, which is weak but not strong for that BRF.

Definition 11. For a given graph G we define the *weak dependency class* with marginal parameter vector $\vec{p} \in [0,1]^V$

$$\mathcal{C}^{\text{weak}}_G(\vec{p}) := \{\text{BRF } Y : \quad \forall\, v \in V : \quad \mathbb{P}(Y_v = 1 | Y_{V\setminus\mathcal{N}_1(v)}) \geq p_v\} \tag{2.3a}$$

and the *strong dependency class* with marginal parameter vector $\vec{p}$

$$\mathcal{C}_G^{\text{strong}}(\vec{p}) := \left\{ \text{BRF } Y : \begin{array}{c} \forall\, v \in V : \quad \mathbb{P}(Y_v = 1) = p_v \\ G \text{ is a strong dependency graph of } Y \end{array} \right\} . \tag{2.3b}$$

They contain the BPF with marginal parameter vector $\vec{p}$ and are thus non-empty. In particular

$$\mathcal{C}_G^{\text{strong}}(\vec{p}) \subseteq \mathcal{C}_G^{\text{weak}}(\vec{p}) . \tag{2.3c}$$

In all but some trivial cases the inclusion 2.3c is strict. The comment after theorem 106 explains, that even demanding equality in (2.3a) does not imply strong independence. On the other hand, the $\geq$ is a sufficiently strong condition to work with. The class of *lop-sided* probability measures [AS08, page 70], [SS05, theorem 1.2] is a subclass of $\mathcal{C}_G^{\text{weak}}(\vec{p})$.

The rest of this section contains the proofs of the statements made above. We first show that the condition of strong dependency graph is as strong as one would like it to be:

Proposition 12. *Let Y with strong dependency graph G and $\{W_i\}_{i\in\mathbb{N}}$ with $W_i \subsetneq V$, $i \neq j \Rightarrow d(W_i, W_j) > 1$, then the $\{Y_{W_i}\}_{i\in\mathbb{N}}$ are independent.*

Proof. Start with $W := W_1$ and $U := \biguplus_{i=2}^{\infty} W_i$. Apply (2.2a) to see their independence. Proceed by induction. □

Furthermore the classes are stable under vertex-wise operations

Proposition 13. *Fix the graph $G := (V, E)$ and a marginal parameter vector $\vec{p} \in [0,1]^V$. Let X and Y be two independent BRFs with marginal parameter vector $\vec{p}$. Let $\diamond$ be a binary operator on $\{0,1\}^2$. Let $Z := X \diamond Y$ with $Z_v := X_v \diamond Y_v$, for every $v \in V$. If X and Y are both pairwise independent, have both weak dependency graph G and have both strong dependency graph G, then Z is pairwise independent, has weak dependency graph G and has strong dependency graph G respectively.*

Proof. It is obvious that all the necessary independent subfields of Z are functions of the same independent subfields of X and Y. Thus they stay independent. □

Example 14 (Pairwise independence does not imply weak dependency)**.** Take the graph $G := ([3], \emptyset)$. For $\varepsilon \in]0, \frac{1}{8}[$, define the $\mathcal{X}_{[3]}$-valued rv Y with distribution

$$\mathbb{P}(Y = \vec{s}) := \begin{cases} \frac{1}{8} + \varepsilon & \text{if } \vec{s} \in \{(0,0,0), (1,0,1), (0,1,1), (1,1,0)\} \\ \frac{1}{8} - \varepsilon & \text{if } \vec{s} \in \{(0,0,1), (0,1,0), (1,0,0), (1,1,1)\} . \end{cases}$$

Thus all Y_i are Bernoulli($\frac{1}{2}$)-distributed. We have $Y_i \perp Y_j$ for $i \neq j$, hence Y is pairwise independent. On the other hand

$$\mathbb{P}(Y = \vec{1}) = \frac{1}{8} + \varepsilon \neq \frac{1}{4}\frac{1}{2} = \mathbb{P}(Y_1 = Y_2 = 1)\mathbb{P}(Y_3 = 1) ,$$

hence $(Y_1, Y_2) \not\perp Y_3$ and Y is not weakly independent on G.

It rests to show that there may not be a unique minimal (strong or weak) dependency graph. The following example is inspired by the corresponding weak example in [SS05, section 4.1].

Example 15 (No unique minimal dependency graph)**.** Let Y be uniformly distributed on

$$A := \{\vec{a} \in \mathcal{X}_{[4]} : \quad \sum_{i=1}^{4} a_i \in \{1, 3\}\}$$

We claim, that the following two graphs are minimal strong dependency graphs of Y. They are not isomorph to each other.

$$G_1 := ([4], \{(1,2), (2,3), (3,4)\})$$
$$G_2 := ([4], \{(1,2), (1,3), (1,4)\})$$

The BRF Y has the following independence properties:

$$Y_1 \perp (Y_2, Y_3) \qquad Y_1 \not\perp (Y_2, Y_3, Y_4) \qquad (Y_1, Y_2) \not\perp (Y_3, Y_4)\,. \tag{2.4}$$

Furthermore the three rvs (Y_1, Y_2, Y_3) are independent. By the symmetry of Y's definition all parts of (2.4) and the triple-independence are invariant under permutations of the indices from [4]. Admitting (2.4) momentarily, the left part of (2.4) and the triple independence imply, that G_1 and G_2 are strong dependency graphs of Y respectively. The middle part and right part of (2.4) forbid the removal of edges from G_1 and G_2 respectively, rendering them minimal.

Regard the quantities

$$p_1 := \mathbb{P}(Y_1 = 1) = \frac{1}{2} \qquad p_{11} := \mathbb{P}(Y_{\{1,2\}} = \vec{1}) = \frac{1}{4}$$
$$p_{111} := \mathbb{P}(Y_{\{1,2,3\}} = \vec{1}) = \frac{1}{8} \qquad p_{1111} := \mathbb{P}(Y = \vec{1}) = \frac{1}{8}$$
$$p_{10} := \mathbb{P}(Y_1 = 1, Y_2 = 0) = \frac{1}{4} \qquad p_{110} := \mathbb{P}(Y_{\{1,2\}} = \vec{1}, Y_3 = 0) = \frac{1}{8}\,.$$

To prove (2.4), we use the permutation invariance of Y. Then the left part follows from $p_{111} = p_{11}p_1$ and $p_{110} = p_{10}p_1$, the middle part from $p_{1111} \neq p_{111}p_1$ and the right part from $p_{1111} \neq p_{11}^2$. Likewise, to show the triple-independence, it is sufficient to verify that $p_{111} = p_1^3$ and $p_{11} = p_1^2$.

2.2 Basics

This section contains a detailed introduction to Shearer's measure on a finite graph. It recapitulates the definition and construction by Shearer in section 2.2.1 and develops its properties in the following sections. New results include the characterization (2.8), an internal coupling in model 22 and the consistent use of OVOEPs (see section 2.2.4).

2.2.1 Definition, construction and characterization

This section describes the construction of *Shearer's signed measure* on a finite graph with given marginal vector, due to [She85]. We define the set of admissible parameters $\mathcal{P}_{sh}^G$ for Shearer's signed measure to be a probability measure. As this case is the prevalent one discussed we omit the qualifier "probability" and just call it *Shearer's measure*. We characterize Shearer's measure in proposition 18.

Model 16. Let $G := (V, E)$ be finite and $\vec{p} \in [0,1]^V$. Define the function $\mu_{G,\vec{p}} : \mathcal{X}_V \to \mathbb{R}$ by

$$\mu_{G,\vec{p}}(Y_W = \vec{0}, Y_{V\setminus W} = \vec{1}) := \sum_{\substack{W \subseteq T \subseteq V\\ T \text{ independent}}} (-1)^{|T|+|W|} \prod_{v\in T} q_v \,. \tag{2.5}$$

Proposition 17. *Model 16 defines a* signed measure $\mu_{G,\vec{p}}$ *on* $\mathcal{X}_V$ *with* total mass 1 *and* strong dependency graph G. *In particular*

$$W \subseteq V \text{ not independent} \Rightarrow \mu_{G,\vec{p}}(Y_W = \vec{0}) = 0\,. \tag{2.6}$$

Proof. The signed measure part is obvious, as we have a real-valued function on $\mathcal{X}_V$. For the total mass we calculate

$$\begin{aligned}
&\sum_{W\subseteq V} \mu_{G,\vec{p}}(Y_W = \vec{0}, Y_{V\setminus W} = \vec{1})\\
&= \sum_{W\subseteq V} \sum_{\substack{W \subseteq T \subseteq V\\ T \text{ independent}}} (-1)^{|T|+|W|} \prod_{v\in T} q_v\\
&= \sum_{T\subseteq V \text{ independent}} (-1)^{|T|} \left(\prod_{v\in T} q_v\right) \underbrace{\sum_{W\subseteq T} (-1)^{|W|}}_{0 \Leftrightarrow T\neq\emptyset}\\
&= 1\,.
\end{aligned}$$

To show that G is a strong dependency graph for Y under $\mu_{G,\vec{p}}$ we regard disjoint $U, U', W, W' \subseteq V$ with $d(U \uplus W, U' \uplus W') > 1$. Then

$$\begin{aligned}
&\mu_{G,\vec{p}}(Y_{W\uplus W'} = \vec{0}, Y_{U\uplus U'} = \vec{1})\\
&= \sum_{\substack{W\uplus W' \subseteq T \subseteq V\setminus(U\uplus U')\\ T \text{ independent}}} (-1)^{|T|+|W|} \left(\prod_{v\in T} q_v\right)\\
&= \sum_{S \text{ independent}} \sum_{\substack{T\uplus T' = S\\ W\subseteq T\subseteq V\setminus U \text{ independent}\\ W'\subseteq T'\subseteq V'\setminus U' \text{ independent}}} (-1)^{|T|+|W|} \left(\prod_{v\in T} q_v\right)\\
&= \left(\sum_{\substack{W \subseteq T \subseteq V\setminus U\\ T \text{ independent}}} (-1)^{|T|+|W|} \prod_{v\in T} q_v\right) \left(\sum_{\substack{W' \subseteq T' \subseteq V\setminus U'\\ T' \text{ independent}}} (-1)^{|T'|+|W'|} \prod_{v\in T'} q_v\right)\\
&= \mu_{G,\vec{p}}(Y_W = \vec{0}, Y_U = \vec{1}) \mu_{G,\vec{p}}(Y_{W'} = \vec{0}, Y_{U'} = \vec{1})\,.
\end{aligned}$$

To show (2.6) we calculate

$$\mu_{G,\vec{p}}(Y_W = \vec{0})$$

$$
\begin{aligned}
&= \sum_{\vec{s}\in\mathcal{X}_{V\setminus W}} \mu_{G,\vec{p}}(Y_W = \vec{0}, Y_{V\setminus W} = \vec{s})\\
&= \sum_{U\subseteq V\setminus W} \mu_{G,\vec{p}}(Y_{W\uplus U} = \vec{0}, Y_{V\setminus(W\uplus U)} = \vec{1})\\
&= \sum_{U\subseteq V\setminus W} \sum_{\substack{W\uplus U\subseteq T\subseteq V\\ T \text{ independent}}} (-1)^{|T|+|W|+|U|} \prod_{v\in T} q_v\\
&= \sum_{\substack{W\subseteq T\subseteq V\\ T \text{ independent}}} (-1)^{|T|+|W|} \left(\prod_{v\in T} q_v\right) \underbrace{\sum_{U\subseteq T\setminus W} (-1)^{|U|}}_{=0 \Leftrightarrow T\setminus W\neq\emptyset \Leftrightarrow T\neq W}\\
&= (-1)^{2|W|} \left(\prod_{v\in W} q_v\right) (-1)^{|\emptyset|}\\
&= \prod_{v\in W} q_v \,.
\end{aligned}
$$

The sum is 0 iff W is not an independent set. □

Define the *set of admissible parameters for Shearer's measure* as

$$
\mathcal{P}_{sh}^{G} := \{\vec{p} \in [0,1]^V : \quad \mu_{G,\vec{p}} \text{ is a probability measure}\} \,. \tag{2.7}
$$

It is easy to see that $\vec{1} \in \mathcal{P}_{sh}^{G}$. Thus it makes sense to talk about $\mathcal{P}_{sh}^{G}$. We denote by $\mathcal{Q}_{G}^{sh} := \vec{1} - \mathcal{P}_{sh}^{G}$ the set of the $\vec{q}$ associated with each $\vec{p} \in \mathcal{P}_{sh}^{G}$. This notation carries over for the later specialisations of $\mathcal{P}_{sh}^{G}$. The uniqueness of $\mu_{G,\vec{p}}$ follows from the following characterization:

Proposition 18. *Let ν be a probability measure on $\mathcal{X}_V$. If Y under ν*

$$
\text{has strong dependency graph } G, \tag{2.8a}
$$

$$
\text{has marginal parameter } \vec{p}, \text{ i.e. } \forall\, v \in V : \quad \mu_{G,\vec{p}}(Y_v = 1) = p_v \,, \tag{2.8b}
$$

$$
\text{and forbids neighbouring } 0\text{s, i.e. } \forall\,(v,w) \in E : \mu_{G,\vec{p}}(Y_{\{v,w\}} = \vec{0}) = 0 \,, \tag{2.8c}
$$

then $\nu = \mu_{G,\vec{p}}$.

Remark. This characterization implies that $\mu_{G,\vec{p}}$ is *stable under projections* onto subgraphs $G(W)$, for $W \subseteq V$. That is for disjoint $U, U' \subseteq W \subseteq V$ we have

$$
\mu_{G,\vec{p}}(Y_U = \vec{0}, Y_{U'} = \vec{1}) = \mu_{G(W),\vec{p}}(Y_U = \vec{0}, Y_{U'} = \vec{1}) \,. \tag{2.9}
$$

Proof. Property (2.8c) is equivalent to (2.6). This allows to construct all other probabilities of ν by the *inclusion-exclusion principle* [Gri10, exercise 2.6] and the use of the independence encoded by G. But the result is just (2.5) – thus the two probability measures coincide. □

2.2.2 The critical function

Define the *critical function* of Shearer's signed measure on G by

$$
\Xi_G : \quad [0,1]^V \to \mathbb{R} \qquad \vec{p} \mapsto \Xi_G(\vec{p}) := \mu_{G,\vec{p}}(Y_V = \vec{1}) = \sum_{\substack{T\subseteq V\\ T \text{ indep}}} \prod_{v\in T} (-q_v) \,. \tag{2.10}
$$

In graph theory (2.10) is also known as the *independent set polynomial* of G [FS90, HL94] and in lattice gas theory as the grand *canonical partition function* at negative fugacity $-\vec{q}$ [SS05, section 1.1].

Proposition 19. *The critical function satisfies a* fundamental identity *(an instance of a* deletion-contraction identity*)*

$$\forall\, v \in V, \vec{p} \in [0,1]^V : \quad \Xi_G(\vec{p}) = \Xi_{G(V \setminus \{v\})}(\vec{p}) - q_v\, \Xi_{G(V \setminus \mathcal{N}_1(v))}(\vec{p}) \,. \tag{2.11}$$

The critical function factorizes over the connected components $(G_i)_{i=1}^n$ *of* G*:*

$$\Xi_G(\vec{p}) = \prod_{i=1}^{n} \Xi_{G_i}(\vec{p}) \,. \tag{2.12}$$

Proof. For (2.11) discriminate between independent sets containing v and those which do not in (2.10). It uses the fact (2.8c) that a 0 in v implies 1s in $\mathcal{N}(v)$. For (2.12) factorize every independent set into a product of independent sets on the connected components. □

The critical functions determine $\mu_{G,\vec{p}}$ and $\mathcal{P}_{sh}^G$:

Proposition 20. *If* $W, U \subseteq V$ *are disjoint and* W *is independent, then*

$$\mu_{G,\vec{p}}(Y_W = \vec{0}, Y_U = \vec{1}) = \left(\prod_{v \in W} q_v \right) \Xi_{G(U \setminus \mathcal{N}_1(W))}(\vec{p}) \,, \tag{2.13}$$

where $\mathcal{N}_1(W) := \biguplus_{w \in W} \mathcal{N}_1(w)$*. This allows us to characterize*

$$\mathcal{P}_{sh}^G = \{\vec{p} \in [0,1]^V : \quad \forall\, W \subseteq V : \quad \Xi_{G(W)}(\vec{p}) \geq 0\} \,. \tag{2.14}$$

Thus $\mathcal{P}_{sh}^G$ *is closed.*

Proof. To show (2.13) we use (2.8c) to calculate

$$\begin{aligned} \mu_{G,\vec{p}}(Y_W = \vec{0}, Y_U = \vec{1}) &= \mu_{G,\vec{p}}(Y_W = \vec{0}, Y_{U \setminus \mathcal{N}_1(W)} = \vec{1}) \\ &= \left(\prod_{v \in W} q_v \right) \Xi_{G(U \setminus \mathcal{N}_1(W))}(\vec{p}) \,. \end{aligned}$$

The characterization (2.14) follows by rewriting (2.7) with the aid of (2.13). The intersection of $[0,1]^V$ with the preimage of $[0,1]$ of a finite number of polynomials is closed. □

Going one step further, we have:

Proposition 21.

$$\forall\, W \subseteq V, v \in V \setminus W, \vec{p} \in \mathcal{P}_{sh}^G : \quad \Xi_{G(W \uplus \{v\})}(\vec{p}) \leq \Xi_{G(W)}(\vec{p}) \,, \tag{2.15}$$

with strict inequality iff $G(W \uplus \{v\})$ *is connected and* $p_v \neq 1$*. Hence*

$$\mathcal{P}_{sh}^G \subseteq \{\vec{p} \in [0,1]^V : \quad \Xi_G(\vec{p}) \geq 0\} \,. \tag{2.16}$$

Remark. (2.16) justifies partly why the critical function is critical. The full picture emerges in sections 2.2.5 and 2.2.3. Equation (2.15) can be extended to also handle deletion of single edges. See [SS05, corollary 2.7], proposition 178 or lemma 117.

Remark. In the light of the projection-stability of $\mu_{G,\vec{p}}$ (2.14) and the criticality (2.16) the characterization (2.14) can be interpreted as "all projections of $\mu_{G,\vec{p}}$ exist".

Proof. We prove (2.15) by induction over the cardinality of W. If $|W| = 0$, then $W = \emptyset$ and we have $\Xi_{G(\{v\})}(\vec{p}) = 1 - q_v \leq 1 = \Xi_{G(\emptyset)}(\vec{p})$, with strict inequality for $p_v < 1$. For the induction step choose suitable W and v. Then we use the fundamental identity (2.11)

$$\Xi_{G(W \uplus \{v\})}(\vec{p}) = \Xi_{G(W)}(\vec{p}) - q_v \Xi_{G(W \setminus \mathcal{N}_1(v))}(\vec{p}) \leq \Xi_{G(W)}(\vec{p}) \,.$$

Equality holds iff $p_v = 1$ or $W \setminus \mathcal{N}_1(v) = W$, that is $G(W \uplus \{v\})$ is disconnected. The characterization (2.16) is a combination of (2.15) and (2.14). □

2.2.3 Couplings

This section describes two couplings involving Shearer's measure. The idea for the first coupling I have to attribute to Pierre Mathieu. It connects Shearer's measure under monotone increasing parameters. We present it in model 22. The idea is to erase 0s in realizations independently of each other. This coupling permits to show certain monotonicity properties of the critical functions and $\mathcal{P}^G_{sh}$ in proposition 24 and propositions 26 and 23 respectively. Finally it allows to characterize $\mathcal{P}^G_{sh}$ by the preimage of Ξ_G in proposition 25. The second coupling already appeared in [She85] and describes what happens outside of $\mathring{\mathcal{P}}^G_{sh}$ in theorem 27. Together with the minimality in theorem 33 in section 2.2.5 it gives an alternate description of $\mathcal{P}^G_{sh}$.

Model 22. Let $\vec{p} \in \mathcal{P}^G_{sh}$ and let Y be $\mu_{G,\vec{p}}$-distributed. Choose a BPF X with parameter $\vec{c}$ independent of Y. Let

$$\vec{r} := \vec{1} - (\vec{1} - \vec{p})(1 - \vec{c}) = \vec{p} + \vec{c} - \vec{p}\vec{c} \,. \tag{2.17}$$

Then $Z := Y \vee X$ (vertex-wise max) is $\mu_{G,\vec{r}}$-distributed.

Proof. We check that Z satisfies the characterization (2.8). By proposition 13 a vertex-wise change by an independent BPF does not change the dependency graph, thus (2.8a) holds. The marginal parameters are

$$1 - r_v = \mathbb{P}(Z_v = 0) = \mathbb{P}(Y_v = 0, X_v = 0) = (1 - p_v)(1 - c_v) \,,$$

whence (2.8b) holds. And (2.8c) follows from the fact, that taking the vertex-wise max with X only flips 0s to 1s, thus all 0s in realizations of Z still form independent sets of G. □

Proposition 23. *The set $\mathcal{P}^G_{sh}$ is an* up-set*. This means, that if $\vec{p} \leq \vec{r}$ and $\vec{p} \in \mathcal{P}^G_{sh}$, then $\vec{r} \in \mathcal{P}^G_{sh}$.*

Proof. Apply the coupling from model 22 with $\vec{c}$ defined by

$$c_v := \begin{cases} 1 - \frac{1-r_v}{1-p_v} & \text{if } p_v < 1 \\ 1 & \text{else.} \end{cases} \tag{2.18}$$

Calculating (2.17) with the $\vec{c}$ from (2.18) results in $\vec{r}$. □

Proposition 24. *The function $\Xi_G(\vec{p})$ is monotone increasing in $\vec{p}$:*

$$\mathcal{P}_{sh}^G \ni \vec{p} \leq \vec{r} \quad \Rightarrow \quad \Xi_G(\vec{p}) \leq \Xi_G(\vec{r}) \,. \tag{2.19}$$

Equality holds iff $\vec{p} = \vec{r}$ or $\Xi_G(\vec{r}) = 0$.

Proof. Using the coupling from model 22, we see that

$$\Xi_G(\vec{p}) \leq \mathbb{P}(X_V = \vec{1})\, \Xi_G(\vec{r}) \,.$$

Unless $\vec{p} = \vec{r}$ or $\Xi_G(\vec{r}) = 0$, we have $\mathbb{P}(X_V = \vec{1}) = \prod_{v \in V} c_v > 0$ and a strict inequality. □

Proposition 25. *The* connected component *of $\vec{1}$ in $[0,1]^V \cap {\Xi_G}^{-1}([0,\infty[)$ equals $\mathcal{P}_{sh}^G$. Thus the* boundary $\partial\mathcal{P}_{sh}^G$ *of $\mathcal{P}_{sh}^G$ is the part of* hypersurface *of $\Xi_G(\vec{p}) = 0$ intersecting $[0,1]^V$ directly visible from $\vec{1}$.*

$$\partial\mathcal{P}_{sh}^G \subseteq \{\vec{p} \in \mathcal{P}_{sh}^G : \quad \Xi_G(\vec{p}) = 0\} \,. \tag{2.20a}$$

The interior $\mathring{\mathcal{P}}_{sh}^G$ *of $\mathcal{P}_{sh}^G$ is*

$$\mathring{\mathcal{P}}_{sh}^G = \{\vec{p} \in \mathcal{P}_{sh}^G : \quad \Xi_G([\vec{p}, \vec{1}]) > 0\} \,. \tag{2.20b}$$

Proof. We know that Ξ_G is multi-affine and thus continuous. We know that $\mathcal{P}_{sh}^G$ contains $\vec{1}$, must be contained in the preimage ${\Xi_G}^{-1}([0,\infty[)$ (by (2.16)), which is closed, and is an up-set (by proposition 23). Combining these facts yields the statement. □

Proposition 26. *The set $\mathcal{P}_{sh}^G$ is monotone non-increasing in G:*

$$W \subseteq U \subseteq V \quad \Rightarrow \quad \mathcal{P}_{sh}^{G(W)} \supseteq \mathcal{P}_{sh}^{G(U)} \,. \tag{2.21}$$

Remark. The statement (2.21) also holds under addition of edges. See proposition 178.

Proof. This is a direct consequence of proposition 25 and the monotonicity of the critical functions in (2.15). □

Theorem 27 ([She85, proof of theorem 1])**.** *Let G be finite. If $\vec{p} \notin \mathring{\mathcal{P}}_{sh}^G$, then there exists a BRF $Z \in \mathcal{C}_G^{strong}(\vec{p})$ with $\mathbb{P}(Z_V = \vec{1}) = 0$.*

Proof. As $\vec{p} \notin \mathring{\mathcal{P}}_{sh}^G$ and $\vec{1} \in \mathring{\mathcal{P}}_{sh}^G$ the line segment $[\vec{p}, \vec{1}]$ crosses $\partial\mathcal{P}_{sh}^G$ at the vector $\vec{r}$ (unique because $\mathring{\mathcal{P}}_{sh}^G$ is an up-set – see proposition 23). Let $\vec{x}$ satisfy the relation $\vec{p} = \vec{x}\,\vec{r}$. Let Y be $\mu_{G,\vec{r}}$-distributed and X be $\Pi_{\vec{x}}^V$-distributed independently of Y. Set $Z := Y \wedge X$. Then $Z \in \mathcal{C}_G^{\text{strong}}(\vec{p})$ and

$$\mathbb{P}(Z_V = \vec{1}) = \mathbb{P}(X_V = \vec{1})\mu_{G,\vec{r}}(Y_V = \vec{1}) = 0 \,.$$

□

2.2.4 One vertex open extension probabilities (OVOEPs)

While the critical functions suffice to describe Shearer's measure, a more nuanced analysis is possible by regarding ratios of critical functions. They are conditional probabilities of the form "open on some vertices | open on some other vertices". The motivation for their investigation stems from their nice interplay with recursive calculations using the fundamental identity (2.23).

Let $v \notin W \subseteq V$. If $\Xi_{G(W)}(\vec{p}) > 0$, then define the *one vertex open extension probability* of (W, v) by

$$\alpha_W^v(\vec{p}) := \mu_{G,\vec{p}}(Y_v = 1 | Y_W = \vec{1})\,. \tag{2.22}$$

Reformulate the *fundamental identity* (2.11) as

$$\alpha_W^v(\vec{p}) = 1 - \frac{q_v}{\prod_{i=1}^m \alpha_{W_i}^{w_i}(\vec{p})}, \tag{2.23}$$

where $W \cap \mathcal{N}(v) =: \{w_1, \dots, w_m\}$ and $W_i := W \setminus \{w_i, \dots, w_m\}$.

Proposition 28. *The fundamental identity* (2.23) *is well-defined.*

Proof. The finiteness of $\alpha_W^v(\vec{p})$ on the lhs of (2.23) implies, that $\Xi_{G(W)}(\vec{p}) > 0$. In turn, this implies by (2.15), that all the terms on the rhs of (2.23) are well-defined, too. □

We can characterize $\mathring{\mathcal{P}}_{sh}^G$ by the OVOEPs:

Proposition 29.

$$\mathring{\mathcal{P}}_{sh}^G = \{\vec{p} \in [0,1]^V : \quad \forall (W, v) : \quad \alpha_W^v(\vec{p}) > 0\}\,. \tag{2.24}$$

Remark. If $\vec{p} \in \partial\mathcal{P}_{sh}^G$, the $\alpha_W^v(\vec{p})$ may become 0 or not even exist (that is, the conditional probability they represent may not be well-defined). This is the reason for working with $\mathring{\mathcal{P}}_{sh}^G$ instead of p_{sh}^G. Those OVOEPs, which are 0 are exactly those pairs (W, v), for which $\vec{p}_{W \uplus \{v\}} \not> \vec{0}$ or $W \uplus \{v\} = V$. Going the other direction, if $\vec{p} \in \mathring{\mathcal{P}}_{sh}^G$, then $\vec{p} > \vec{0}$.

Proof. We show the equivalence of (2.14) and (2.24).

(2.14) ⊇ (2.24): For all $W \subseteq V$ the $\Xi_{G(W)}(\vec{p})$ are positive, thus their ratios are well-defined and positive, too.

(2.14) ⊆ (2.24): For all $\{w_1, \dots, w_m\} := W \subseteq V$ let $W_i := \{w_1, \dots, w_{i-1}\}$. As all the $\alpha_{W_i}^{w_i}(\vec{p})$ exists and are positive, we factorize

$$\Xi_{G(W)}(\vec{p}) = \prod_{i=1}^m \alpha_{W_i}^{w_i}(\vec{p}) > 0\,.$$

□

Remark. A similar approach (see [SS05, section 5.3]) is to define $\mathcal{P}_{sh}^G$ by the well-definedness of quantities obeying the fundamental identity (2.23). For fixed $\vec{p}$, define $a(W,v)$ by $a(\emptyset,v) := p_v$ and the recursion $a(W,v) := 1 - \frac{q_v}{a(W_i,w_i)}$ (with the same notation as in the fundamental identity (2.23)). Then $\vec{p} \in \mathcal{P}_{sh}^G$ iff all $a(W,v)$ are well-defined and non-negative. This approach lacked the interpretation of the $a(W,v)$ as conditional probabilities, though.

Definition 30. Call the pair (W,v), respectively α_W^v, *escaping* iff $\mathcal{N}(v) \setminus W \neq \emptyset$ and call every vertex $w \in \mathcal{N}(v) \setminus W$ an *escape* of (W,v).

The notion of *escaping pair* is inspired by the proof of [She85, theorem 2]. It allows us to push the probability of encountering a 0 instead of a 1 at v away. This yields a lower bound on $\alpha_W^v(\vec{p})$ independent of W (except for the condition $w \notin W$).

Proposition 31. *Let $\vec{p} \in \mathcal{P}_{sh}^G$, then*

$$\forall\,(W,v): \quad \alpha_W^v(\vec{p}) \leq p_v \tag{2.25a}$$

and

$$\forall\,(W,v), w \in \mathcal{N}(v) \setminus W: \quad q_w \leq \alpha_W^v(\vec{p}). \tag{2.25b}$$

Proof. The fundamental identity (2.23) implies, that

$$\alpha_W^v(\vec{p}) = 1 - \frac{q_v}{\prod \alpha_\star^\star(\vec{p})} \leq 1 - q_v = p_v\,.$$

Likewise, if (W,v) is escaping with escape $w \in \mathcal{N}(v) \setminus W$, then (2.23) yields

$$0 \leq \alpha_{W \uplus \{v\}}^w(\vec{p}) = 1 - \frac{q_w}{\alpha_W^v(\vec{p}) \prod \alpha_\star^\star(\vec{p})} \leq 1 - \frac{q_w}{\alpha_W^v(\vec{p})}\,.$$

□

Proposition 32. *Let $\vec{p} \in \mathcal{P}_{sh}^G$. The OVOEPs are monotone non-increasing in W:*

$$W \subseteq U \quad \Rightarrow \quad \alpha_W^v(\vec{p}) \geq \alpha_U^v(\vec{p})\,. \tag{2.26a}$$

The inequality is strict iff the connected component of v in $G(W \uplus v)$ increases in $G(U \uplus v)$ and $p_v < 1$. The OVOEPs are monotone non-decreasing in $\vec{p}$:

$$\vec{p} \leq \vec{r} \quad \Rightarrow \quad \alpha_W^v(\vec{p}) \leq \alpha_W^v(\vec{r})\,. \tag{2.26b}$$

The inquality is strict if $\vec{r} - \vec{p}$ is non-zero in the connected component of v in $G(W \uplus v)$. In particular

$$\lim_{p_v \to 1} \alpha_W^v(\vec{p}) = 1\,. \tag{2.26c}$$

Remark. The OVOEP α_W^v also decreases when adding edges to $G(W)v$. See proposition 178.

Proof. We assume throughout the proof, that all the OVOEPs are well-defined. The strict results follow directly from the proof if the increase by U neighbours v or $p_v < r_v$. Otherwise it is so for some other vertex in the connected component of v and carry the strictness forward through the applications of the fundamental identity.

(2.26a): We prove the statement by induction over the cardinality of W, simultaneously for all v. The induction base with $W = \emptyset$ is

$$\alpha^v_\emptyset(\vec{p}) = 1 - q_v \begin{cases} \geq \frac{1-q_v-q_w}{1-q_w} = \alpha^v_{\{w\}}(\vec{p}) & \text{if } v \backsim w \\ = 1 - q_v = \alpha^v_{\{w\}}(\vec{p}) & \text{if } v \not\backsim w\,. \end{cases}$$

For the induction step we add just one vertex w to W and set $U := W \uplus \{w\}$. Let $\{w_1, \ldots, w_m\} := \mathcal{N}(v) \cap U$. First assume that $w \not\backsim v$. Using the fundamental identity (2.23) we have

$$\alpha^v_U(\vec{p}) = 1 - \frac{q_v}{\prod_{i=1}^m \alpha^{w_i}_{U \setminus \{w_i,\ldots,w_m\}}(\vec{p})} \leq 1 - \frac{q_v}{\prod_{i=1}^m \alpha^{w_i}_{W \setminus \{w_i,\ldots,w_m\}}(\vec{p})} = \alpha^v_W(\vec{p})\,.$$

Secondly assume that $v \backsim w = w_m$. Hence the fundamental identity (2.23) yields

$$\alpha^v_U(\vec{p}) = 1 - \frac{q_v}{\prod_{i=1}^m \alpha^{w_i}_{U \setminus \{w_i,\ldots,w_m\}}(\vec{p})} < 1 - \frac{q_v}{\prod_{i=1}^{m-1} \alpha^{w_i}_{W \setminus \{w_i,\ldots,w_{m-1}\}}(\vec{p})} = \alpha^v_W(\vec{p})\,.$$

(2.26b): We prove (2.26b) by induction over the cardinality of W, simultaneously for all v. The induction base with $W = \emptyset$ is given by

$$\alpha^v_\emptyset(\vec{p}) = \Xi_{G(\{v\})}(\vec{p}) = 1 - p_v \leq 1 - r_v = \Xi_{G(\{v\})}(\vec{r}) = \alpha^v_\emptyset(\vec{r})\,.$$

For the induction step we reuse the notation from (2.11): $\{w_1, \ldots, w_m\} := W \cap \mathcal{N}(v)$ and $W_i := W \setminus \{w_i, \ldots, w_m\}$. Then

$$\begin{aligned} &\alpha^v_W(\vec{p}) \\ &= 1 - \frac{1-p_v}{\prod_{i=1}^m \alpha^{w_i}_{W_i}(\vec{p})} && \text{by the fundamental identity (2.11)} \\ &= 1 - \frac{1-r_v}{\prod_{i=1}^m \alpha^{w_i}_{W_i}(\vec{p})} && \text{as } p_v \leq r_v \\ &= 1 - \frac{1-r_v}{\prod_{i=1}^m \alpha^{w_i}_{W_i}(\vec{r})} && \text{by the induction hypothesis as } |W_i| < |W| \\ &= \alpha^v_W(\vec{r}) && \text{by the fundamental identity (2.11).} \end{aligned}$$

(2.26c): Fix (W, v). We deal with a parameter increase only in v, that is $\vec{r} = \vec{p} + (r_v - p_v)\mathbb{I}_v$. Choose $\vec{c}$ as in (2.18), then the coupling from model 22 yields

$$\alpha^v_W(\vec{r}) = \alpha^v_W(\vec{p}) + c_v(1 - \alpha^v_W(\vec{p})) \geq \alpha^v_W(\vec{p})\,. \tag{2.27}$$

The inequality is strict iff $c_v = 0$, that is if $p_v = r_v$, or $\alpha^v_W(\vec{p}) = 1$, that is $p_v = 1$. If we regard $\vec{p}$ with $p_v < 1$ and let $r_v \to 1$, then $c_v \to 1$ and $\alpha^v_W(\vec{r}) \to 1$. □

2.2.5 Minimality

The importance of Shearer's measure is due to its *uniform minimality* with respect to certain conditional probabilities. This minimality is one half of the full picture for Shearer's measure, the other half is given in theorem 27 in section 2.2.3.

Theorem 33 ([She85, theorem 1])**.** *Let* $\nu \in \mathcal{C}_G^{weak}(\vec{p})$ *be a probability measure. If* $\vec{p} \in \mathring{\mathcal{P}}_{sh}^G$ *and* $W \subseteq U \subseteq V$*, then*

$$0 < \mu_{G,\vec{p}}(Y_U = \vec{1} | Y_W = \vec{1}) \leq \nu(Y_U = \vec{1} | Y_W = \vec{1}) \,. \tag{2.28a}$$

In particular, we have for $\vec{p} \in \mathcal{P}_{sh}^G$ *and* $W \subseteq V$*:*

$$0 \leq \Xi_{G(W)}(\vec{p}) \leq \nu(Y_W = \vec{1}) \,. \tag{2.28b}$$

Remark. The LLL is independent of the cardinality of V, that is the finiteness of G.

Proof. It is sufficient to show (2.28a) for the one vertex extensions with $U := W \uplus \{v\}$ and $v \in V \setminus W$. Analoguously to (2.22) and subject to the same conditions, we can define one vertex open extension probabilites for ν:

$$\beta_W^v := \nu(Y_v = 1 | Y_W = \vec{1}) \,. \tag{2.29}$$

Strong independence of Y under ν with respect to G implies that

$$\begin{aligned} \nu(Y_v = 1, Y_W = \vec{1}) &= \nu(Y_W = \vec{1}) - \nu(Y_v = 0, Y_W = \vec{1}) \\ &\geq \nu(Y_W = \vec{1}) - \nu(Y_v = 0)\nu(Y_{W \setminus \mathcal{N}(v)} = \vec{1}) \,. \end{aligned}$$

This lets us state *fundamental inequality* for ν (using the notation from (2.23))

$$\beta_W^v \geq 1 - \frac{q_v}{\prod_{i=1}^m \beta_{W_i}^{w_i}} \,. \tag{2.30}$$

Thus (2.28a) reduces to showing the well-definedness of the inequality

$$\forall \, (W, v) : \quad 0 < \alpha_W^v(\vec{p}) \leq \beta_W^v \,. \tag{2.31}$$

We show (2.31) by induction over the cardinality of W, simultaneously for all v. The induction base with $W = \emptyset$ is

$$\alpha_\emptyset^v = \Xi_{(\{v\}, \emptyset)}(\vec{p}) = p_v = \nu(Y_v) = \beta_\emptyset^v \,.$$

For the induction step we reuse the notation from (2.11): $\{w_1, \dots, w_m\} := W \cap \mathcal{N}(v)$ and $W_i := W \setminus \{w_i, \dots, w_m\}$. Then

$$\begin{aligned} & 0 \\ & < \alpha_W^v(\vec{p}) && \text{as } \vec{p} \in \mathring{\mathcal{P}}_{sh}^G \\ & = 1 - \frac{q_v}{\prod_{i=1}^m \alpha_{W_i}^{w_i}(\vec{p})} && \text{by the fundamental identity (2.11)} \\ & \leq 1 - \frac{q_v}{\prod_{i=1}^m \beta_{W_i}^{w_i}} && \text{by the induction hypothesis as } |W_i| < |W| \\ & \leq \beta_W^v && \text{by the fundamental inequality (2.30)} \,. \end{aligned}$$

To show (2.28b) we have to differentiate between two cases. In the first case $\Xi_{G(W)}(\vec{p}) > 0$. We let $\{w_1, \dots, w_n\} := W$ and $W_i := \{w_1, \dots, w_i\}$ and factorize to apply (2.28a):

$$0 < \Xi_{G(W)}(\vec{p}) = \prod_{i=1}^n \alpha_{W_i}^{v_i}(\vec{p}) \leq \prod_{i=1}^n \beta_{W_i}^{v_i} = \nu(Y_W = \vec{1}) \,.$$

In the second case $\Xi_{G(W)}(\vec{p}) = 0$, which is always a lower bound. □

2.2.6 The Lovász Local Lemma

In this section we state and prove the classic Lovász Local Lemma [EL75]. It shows that $\mathcal{P}_{sh}^{G}$ has non-trivial volume. Its *symmetric version* is corollary 39 in section 2.4.

Theorem 34 ([EL75])**.** *If there exists a vector* $\vec{s} \in]0,1[^{V}$*, such that*

$$\forall\, v \in V : \quad q_v \leq (1 - s_v) \prod_{w \in \mathcal{N}(v)} s_w \,, \tag{2.32a}$$

then

$$\forall\, (W, v) : \quad \alpha_W^v(\vec{p}) \geq s_v > 0 \,. \tag{2.32b}$$

In particular $\vec{p} \in \mathring{\mathcal{P}}_{sh}^{G}$ *and*

$$\Xi_G(\vec{p}) \geq \prod_{v \in V} s_v > 0 \,. \tag{2.32c}$$

Proof. The claim (2.32c) follows directly from (2.32b) by successive conditioning. We prove the claim (2.32b) by induction over the cardinality of W, simultaneously for all v. The induction base with $W = \emptyset$ is

$$\alpha_{\emptyset}^{v} = \Xi_{(\{v\},\emptyset)}(\vec{p}) = p_v = 1 - q_v \geq 1 - (1 - s_v) = s_v \,.$$

For the induction step we reuse the notation from (2.11): $\{w_1, \dots, w_m\} := W \cap \mathcal{N}(v)$ and $W_i := W \setminus \{w_i, \dots, w_m\}$. Then

$$\begin{aligned}
&\alpha_W^v(\vec{p}) \\
&= 1 - \frac{q_v}{\prod_{i=1}^{m} \alpha_{W_i}^{w_i}(\vec{p})} && \text{by the fundamental identity (2.11)} \\
&\geq 1 - \frac{q_v}{\prod_{i=1}^{m} s_{w_i}} && \text{by the induction hypothesis as } |W_i| < |W| \\
&\geq 1 - \frac{q_v}{\prod_{w \in \mathcal{N}(v)} s_w} && \text{taking all neighbours} \\
&\geq 1 - (1 - s_v) && \text{applying condition (2.32a)} \\
&= s_v \,.
\end{aligned}$$

□

2.2.7 Non-markovianity

Shearer's measure is *non-markovian.* This is demonstrated by the minimal example 35, which is contained in every non-trivial graph G. We use homogeneous parameters only for convenience. A more general statement using OVOEPs is in proposition 36.

Example 35. Let $G := ([3], (1,2), (2,3))$ and let $p_{sh}^{G} \leq p < 1$. Then

$$\mu_{G,p}(\vec{s}) = \begin{cases} q & \text{if } \vec{s} \in \{(0,1,1), (1,0,1), (1,1,0)\} \\ q^2 & \text{if } \vec{s} = (0,1,0) \\ 1 - 3q + q^2 & \text{if } \vec{s} = (1,1,1) \\ 0 & \text{else.} \end{cases}$$

Therefore

$$\mu_{G,p}(0_1|1_2)\mu_{G,p}(0_3|1_2) = \left(\frac{q}{1-q}\right)^2 \neq \frac{q^2}{1-q} = \mu_{G,p}(0_1 0_3|1_2)$$

or

$$\mu_{G,p}(1_1|0_2)\mu_{G,p}(1_3|0_2) = \left(\frac{q}{1-q}\right)^2 \neq \frac{q}{1-q} = \mu_{G,p}(1_1 1_3|0_2)\,.$$

Either one implies, that $\mu_{G,p}$ is non-markovian.

Proposition 36. *Let $G := (V, E)$ be a finite graph. Let A, B, C be three non-empty and disjoint subsets of V with $d(A, B) \geq 1$ and $G(A \uplus B \uplus C)$ connected. Let $\vec{1} > \vec{p} \in \mathring{\mathcal{P}}_{sh}^G$. Then*

$$\mu_{G,\vec{p}}(Y_A = \vec{1}, Y_B = \vec{1}|Y_C = \vec{1}) < \mu_{G,\vec{p}}(Y_A = \vec{1}|Y_C = \vec{1})\mu_{G,\vec{p}}(Y_B = \vec{1}|Y_C = \vec{1})\,. \tag{2.33}$$

Thus $\mu_{G,\vec{p}}$ is not markovian.

Proof. Enumerating $\{a_1, \dots, a_n\} := A$ and using the strict version of (2.26a), we have

$$\begin{aligned}
&\mu_{G,\vec{p}}(Y_A = \vec{1}, Y_B = \vec{1}|Y_C = \vec{1})\\
&= \mu_{G,\vec{p}}(Y_A = \vec{1}|Y_B = \vec{1}, Y_C = \vec{1})\mu_{G,\vec{p}}(Y_B = \vec{1}|Y_C = \vec{1})\\
&= \left(\prod_{i=1}^{n} \alpha^{a_i}_{C\uplus B\uplus\{a_1,\dots,a_{i-1}\}}(\vec{p})\right)\mu_{G,\vec{p}}(Y_B = \vec{1}|Y_C = \vec{1})\\
&< \left(\prod_{i=1}^{n} \alpha^{a_i}_{C\uplus\{a_1,\dots,a_{i-1}\}}(\vec{p})\right)\mu_{G,\vec{p}}(Y_B = \vec{1}|Y_C = \vec{1})\\
&= \mu_{G,\vec{p}}(Y_A = \vec{1}|Y_C = \vec{1})\mu_{G,\vec{p}}(Y_B = \vec{1}|Y_C = \vec{1})\,.
\end{aligned}$$

□

2.3 Going infinite

This section shows the existence of Shearer's measure on an infinite graph, as well as its uniqueness and characterization, in theorem 37. Various related limit quantities are discussed, in particular the limit of OVOEPs in (2.35).

If G is *infinite*, then we define

$$\mathcal{P}_{sh}^G := \bigcap_{W\subseteq V, |W|<\infty} \mathcal{P}_{sh}^{G(W)} = \bigcap_{E'\subseteq E, |E'|<\infty} \mathcal{P}_{sh}^{(V,E')} \tag{2.34a}$$

$$\mathring{\mathcal{P}}_{sh}^G := \bigcap_{W\subseteq V, |W|<\infty} \mathring{\mathcal{P}}_{sh}^{G(W)} = \bigcap_{E'\subseteq E, |E'|<\infty} \mathring{\mathcal{P}}_{sh}^{(V,E')}\,. \tag{2.34b}$$

This is well defined by the monotonicity of $\mathcal{P}_{sh}^{\cdot}$ for finite graphs (2.21). The set $\mathring{\mathcal{P}}_{sh}^G$ is *not the interior* of the closed set $\mathcal{P}_{sh}^G$. See (2.39) for an example of a boundary point of $\mathring{\mathcal{P}}^G_{sh}$. The equalities on the rhs of (2.34) are a consequence of the comment after (2.15). It is stated here only for completeness and a justification can be derived from [SS05, corollary 2.7].

We justify the definition of $\mathcal{P}^G_{sh}$ by the following theorem:

Theorem 37. *Let $\vec{p} \in \mathcal{P}^G_{sh}$. Then $\mu_{G,\vec{p}}$ exists, is unique and characterized by* (2.8).

Remark. The motivation for the introduction of $\mathring{\mathcal{P}}^G_{sh}$ is to get rid of all of the degenerate cases, in particular with respect to the well-definedness of OVOEPs.

Remark. The coupling from model 22 also holds in the infinite case.

Proof. Existence: For $\vec{p} \in \mathcal{P}^G_{sh}$, the family $\{\mu_{G(W),\vec{p}} : W \Subset V\}$ forms a consistent family à la Kolmogorov [Bil95, (36.1) & (36.2)]. Hence Kolmogorov's existence theorem [Bil95, theorem 36.2] establishes the existence of an extension of this family, which we call $\mu_{G,\vec{p}}$.

Uniqueness: Let Y be $\mu_{G,\vec{p}}$-distributed. We see that

$$\{\omega \in \Omega : \quad Y_W(\omega) = \mathbb{I}_U\}_{U \subset W \Subset V}$$

is an *intersection stable ring* (π–ring) generating $\sigma(Y)$. Thus uniqueness follows from the π-λ theorem [Bil95, theorem 3.3].

Characterization: The extension $\mu_{G,\vec{p}}$ satisfies all the properties (2.8) on the infinite graph G. The properties are all extensive and thus inherited from the consistent family.

Conversely let ν be a probability measure having the properties (2.8). Then all its finite marginals have them, too, and they coincide with Shearer's measure. Hence by the uniqueness of the Kolmogorov extension ν coincides with $\mu_{G,\vec{p}}$ and (2.8) *characterizes* $\mu_{G,\vec{p}}$ also on infinite graphs. ☐

It is convenient to define the limit of OVOEPs. Let U be an infinite subset of V and $v \notin U$, then

$$\alpha^v_U(\vec{p}) := \inf\{\alpha^v_W(\vec{p}) : \quad v \notin W \Subset U\}\,. \tag{2.35}$$

Proposition 38. *We have*

$$\alpha^v_U(\vec{p}) = \lim_{W \nearrow U} \alpha^v_W(\vec{p})\,. \tag{2.36}$$

Also $\alpha^v_U(\vec{p})$ is monotone decreasing in U and monotone increasing in $\vec{p}$.

Remark. If the quantities involved in the rhs of the fundamental identity (2.23) are non-zero, then (2.36) asserts that the fundamental identity (2.23) also holds for the corresponding infinite versions of the OVOEPs. That is, let $U \cap \mathcal{N}(v) =: \{w_1, \ldots, w_m\}$ and $U_i := U \setminus \{w_m, \ldots, w_i\}$, then

$$\alpha^v_U(\vec{p}) = 1 - \frac{q_v}{\prod_{i=1}^m \alpha^{w_i}_{U_i}(\vec{p})}\,. \tag{2.37}$$

Proof. This follows from the monotonicity of α^v_W in W (2.26a). Taking the limit along a growing sequence of finite sets $(W_n)_{n\in\mathbb{N}}$ exhausting U yields (2.36). The monotonicity properties of $\alpha^v_U(\vec{p})$ itself are inherited from the ones for $\alpha^v_W(\vec{p})$ (see (2.26a) and (2.26b)). □

2.4 In the homogeneous case

Recall that in the homogeneous case we write p in lieu of the vector $p\vec{1}$. The only major difference is, that a single value, the *critical parameter*, describes the relevant *cross-section* of the set of admissible parameters $\mathcal{P}^G_{sh}$. In the case of a finite graph G define the critical parameter p^G_{sh} by

$$p^G_{sh} := \max\{p : \Xi_G(p) \leq 0\} = \min\{p : \mu_{G,p} \text{ is a probability measure}\}\,. \tag{2.38a}$$

In the case of an infinite graph G this becomes

$$p^G_{sh} := \sup\{p^H_{sh} : H \text{ finite subgraph of } G\}\,. \tag{2.38b}$$

Remark. In the infinite case $p^G_{sh}\vec{1}$ is a boundary point of the *non-open set* $\mathring{\mathcal{P}}^G_{sh}$:

$$[p^G_{sh}, 1] = \bigcap_{W \Subset V}]p^{G(W)}_{sh}, 1]\,. \tag{2.39}$$

The critical values are monotone increasing in G. Thus we actually have

$$p^G_{sh} = \lim_{W \nearrow V} p^{G(W)}_{sh}\,. \tag{2.40}$$

In the homogeneous case we have the so-called *symmetric* version of the Lovász Local Lemma as a corollary of theorem 34:

Corollary 39 ([EL75])**.** *If G is uniformly bounded with degree D, then*

$$p^G_{sh} \leq 1 - \frac{D^D}{(D+1)^{(D+1)}}\,. \tag{2.41}$$

Proof. Let $a_v := \frac{D}{D+1}$ and apply theorem 34. □

There is an improved version of the symmetric LLL, which has been discovered independently several times. Its inhomogeneous generalization is proposition 48.

Theorem 40 ([LSS97, theorem 1.3], [SS05, corollary 5.7])**.** *If G is uniformly bounded with degree D, then*

$$p^G_{sh} \leq 1 - \frac{(D-1)^{(D-1)}}{D^D}\,. \tag{2.42a}$$

In particular, for escaping OVOEPs, we have

$$\forall\,(W,v)\ \textit{escaping}: \quad \alpha^v_W(p) \geq 1 - \frac{1}{D}\,. \tag{2.42b}$$

Proof. Assume that $q \leq \frac{(D-1)^{(D-1)}}{D^D}$. Then (2.42b) implies that $\Xi_{G(W)}(p) \geq \left(\frac{D-1}{D}\right)^{|W|} > 0$, for every finite $W \subseteq V$. Hence $p \geq p_{sh}^G$. We prove (2.42b) by induction over the cardinality of W, simultaneously for all v. The induction base is given by

$$\alpha_\emptyset^v(p) = p \geq 1 - \frac{(D-1)^{(D-1)}}{D^D} \geq 1 - \frac{1}{D}.$$

As (W, v) is escaping v has at most $m \leq D-1$ neighbours in W, which we denote by $\{w_1, \ldots, w_m\} := W \cap \mathcal{N}(v)$. Using the fundamental identity (2.23) and (2.42b) the induction step is

$$\alpha_W^v(p) = 1 - \frac{q}{\prod_{i=1}^m \alpha_{W \setminus \{w_i, \ldots, w_m\}}^{w_i}(p)}$$
$$\geq 1 - \frac{q}{\prod_{i=1}^m (1 - \frac{1}{D})} \geq 1 - \frac{q}{\left(\frac{D-1}{D}\right)^{D-1}} \geq 1 - \frac{1}{D}.$$

The key point is, that the OVOEPs on the rhs of an application of the fundamental identity to an escaping OVOEP are always escaping themselves with escape v. □

2.5 First examples and sufficient conditions

This section gives some examples of p_{sh}^G and $\mu_{G,p}$ for particular finite graphs in section 2.5.1. We state the known improvements on the LLL, that is sufficient conditions for $\vec{p} \in \mathcal{P}_{sh}^G$, in section 2.5.2.

2.5.1 Simple examples

This section gives some simple examples on very simple graphs and closes with a converse of the symmetric LLL from corollary 39 in proposition 45.

Example 41. Shearer's measure on a line of n points, L_n,

n	$\Xi_{L_n}(p)$	$q_{sh}^{L_n}$
1	$1-q$	1
2	$1-2q$	$\frac{1}{2} = 0.5$
3	$1-3q+q^2$	$\frac{3-\sqrt{5}}{2} = 0.381966$
4	$1-4q+3q^2 = (1-3q)(1-q)$	$\frac{1}{3} = 0.33333$
5	$1-5q+6q^2-q^3$	...

Example 42. Shearer's measure on a circle C_n of n points

n	$\Xi_{C_n}(p)$	$q_{sh}^{C_n}$
1	$1-q$	1
2	$1-2q$	$\frac{1}{2} = 0.5$
3	$1-3q$	$\frac{1}{3} = 0.3333$
4	$1-4q+2q^2$	$1 - \frac{1}{\sqrt{2}} = 0.29289$
5	$1-5q+5q^2$	$\frac{1}{2} - \frac{1}{2\sqrt{5}} = 0.276393$

Example 43. Shearer's measure on a *star* S_n with one central point o and n leaves. By [SS05], example 3.4, we get $\Xi_{S_n}(p) = (1-q)^n - q$. The measure can be decomposed in a two step fashion: If $Y_o = 0$, then all leaves realize in 1, otherwise $Y_o = 1$ and the leaves are iid Bernoulli(p).

Proposition 44. *Let S_n be as in example 43, then*

$$\lim_{n\to\infty} p_{sh}^{S_n} = 1. \tag{2.43}$$

Proof. $p_{sh}^{S_n}$ is defined as the solution of $(1-q)^n = q$ (see example 43). Excluding the non-solution $q = 0$ we see that $\forall\, \varepsilon > 0 : \exists\, N \in \mathbb{N} : \forall\, n \geq N : (1-q)^n < \varepsilon$, from which we deduce that $q_{sh}^{S_n} \xrightarrow[n\to\infty]{} 0$. □

Proposition 45. *Let G be a graph. Then $p_{sh}^G < 1$ iff G is uniformly bounded.*

Proof. In light of the symmetric LLL in corollary 39 we only need to show the forward implication. If G is not uniformly bounded, then it contains stars S_n for arbitrary $n \in \mathbb{N}$. Using (2.43) we get $1 = \lim_{n\to\infty} p_{sh}^{S_n} \leq p_{sh}^G \leq 1$. □

Example 46. Shearer's measure on the *complete graph* on n points, K_n. By [SS05], example 3.1, we get $\Xi_{K_n}(p) = 1 - nq$ and $p_{sh}^{K_n} = 1 - \frac{1}{n}$. Here the measure consists of $n+1$ events: $\mu_{K_n,p}(Y_V = \vec{1}) = 1 - nq$ and $\forall\, v \in V : \mu_{K_n,p}(Y_v = 0, Y_{V\setminus\{v\}} = \vec{1}) = q$.

Proposition 47. *Let K_n be as in example 46, then*

$$\lim_{n\to\infty} p_{sh}^{K_n} = \lim_{n\to\infty} 1 - \frac{1}{n} = 1. \tag{2.44}$$

2.5.2 Improved versions of the Lovász Local Lemma

This section states improved versions of the LLL, with references to the proofs. We start with the inhomogeneous version of the improved symmetric LLL from theorem 40:

Proposition 48. *Let $\vec{p} \in [0,1]^V$. If there exists $\vec{s} \in]0,\infty[^V$, such that*

$$\forall\, v \in V : \quad q_v(1+s_v) \max\left\{ \prod_{u \in \mathcal{N}(v)\setminus\{w\}} (1+s_u) : \quad w \in \mathcal{N}(v) \right\} \leq s_v\,, \tag{2.45a}$$

then $\vec{p} \in \mathring{\mathcal{P}}_{sh}^G$. In particular

$$\forall\, (W,v) \text{ escaping} : \quad \alpha_W^v(\vec{p}) \geq \frac{1}{1+s_v} > 0\,. \tag{2.45b}$$

Remark. The condition improves upon the LLL (2.32a) and is tight on infinite trees. See section 5.6.6.

Proof. See proposition 123. □

Fernández & Procacci derived another more recent and elaborate sufficient condition for a vector $\vec{p}$ to lie in $\mathring{\mathcal{P}}_{sh}^G$:

Theorem 49 ([FP07, theorem 1],[BFPS11])**.** *Let $\vec{p} \in [0,1]^V$. If there exists $\vec{s} \in]0,\infty[^V$, such that*

$$\forall\, v \in V : \quad q_v\, \Xi_{G(\mathcal{N}_1(v))}(-\vec{s}) \leq s_v\,, \tag{2.46a}$$

then $\vec{p} \in \mathring{\mathcal{P}}^G_{sh}$. In particular

$$\forall\, (W,v) : \quad \alpha^v_W(\vec{p}) \geq (1-q_v)^{\Xi_{G(\mathcal{N}_1(v))}(-\vec{s})-1}\,. \tag{2.46b}$$

The minus in (2.46a) stems from their cluster expansion technique and assures that $\Xi_{G(\mathcal{N}_1(v))}(-\vec{s}) \geq 1 + s_v$, whence $q_v \leq 1$. The condition strictly improves upon the LLL (2.32a) in the case of graphs containing triangles, which $\Xi_{G(\mathcal{N}_1(v))}$ takes into account.

Further sufficient conditions are in chapter 5. A necessary condition via pruned SAW-trees is in theorem 180.

2.6 Quasi-transitive tree-like graphs

This section presents several examples of infinite graphs, notably the k-fuzz of $\mathbb{Z}$ in section 2.6.3 and the d-regular tree in section 2.6.4. For these examples we calculate p^G_{sh} and construct $\mu_{G,p}$ explicitly for all $p \in [p^G_{sh}, 1]$. The construction is based on a generic *zero-one switch* presented in section 2.6.1. The common property of these graphs is that they are quasi-transitive and tree-like. Quasi-transitiveness enables us to shift OVOEPs around in the graph. This is used in conjunction with the tree-likeness to ground the recursions of the fundamental identity. In the limit we get equations, whose solutions determine both p^G_{sh} and $\mu_{G,p}$. We terminate with a conjecture for the whole class in section 2.6.6.

2.6.1 Intrinsic construction via OVOEPs

This section shows how to construct Shearer's measure from a BPF with parameters given by suitable OVOEPs. In full generality, the construction shown is rather unhandy, but it illustrates two principles. First, a generic version of the *zero-one switch* employed in the constructions for regular trees in model 55 and the k-fuzz of $\mathbb{Z}$ in model 55. Second, as a close approximation to intrinsic stochastic domination result in proposition 112.

Model 50. Let $G := (V,E)$ be a connected graph. Let $S_v \subseteq \mathcal{N}(v)$ and $\vec{p} \in \mathring{\mathcal{P}}^G_{sh}$. Let $X := (X_v)_{v\in V}$ be a BPF with $\mathbb{P}(X_v = 1) := \alpha^v_{V\setminus\{v\}}(\vec{p})$. Define $Y := (Y_v)_{v\in V}$ by

$$\forall\, v \in V : \quad Y_v := 1 - (1-X_v) \prod_{w\in S_w} X_w\,. \tag{2.47}$$

Proposition 51. *Let Y be the BRF from model 50. If*

$$\forall\, (v,w) \notin E : \quad S_v \cap S_w = \emptyset \tag{2.48a}$$

$$\forall\, (v,w) \in E : \quad v \in S_w \vee v \in S_v\,, \tag{2.48b}$$

then Y is $\mu_{G,\vec{r}}$-distributed, with $\vec{r} \geq \vec{p}$.

Remark. In (2.48a), it suffices, that if $(v,w) \in E$, that either $v \in S_w$ or $w \in S_v$. The conditions (2.48) are easy to fulfil on tree-like graphs, with better parameters. The choice of $\alpha^v_{V\setminus\{v\}}$ as parameter amounts to ignoring particular structural properties of the graph.

Proof. We check that Y satisfies the characterization (2.8). We get the estimate on the marginal parameter $\vec{r}$ by:

$$\begin{aligned}
&1 - r_v \\
&= \mathbb{P}(Y_v = 0) && \text{by the construction (2.47)} \\
&= \mathbb{P}(X_v = 0, X_{S_v} = \vec{1}) \\
&= (1 - \alpha^v_{V\setminus\{v\}}(\vec{p})) \prod_{w \in S_v} \alpha^w_{V\setminus\{w\}}(\vec{p}) && \text{by the independence of } X \\
&= (1 - \alpha^v_{V\setminus\{v\}}(\vec{p})) \prod_{j=1}^{|S_v|} \alpha^{w_j}_{V\setminus\{w_j\}}(\vec{p}) && \text{ordering } S_v \\
&\le (1 - \alpha^v_{V\setminus\{v\}}(\vec{p})) \prod_{j=1}^{|S_n|} \alpha^{w_j}_{V\setminus\{v, w_{|S_n|}, \dots, w_j\}}(\vec{p}) && \text{monotonicity from (2.26a)} \\
&= 1 - p_v && \text{by the limit of (2.23)}\,.
\end{aligned}$$

Let $(v,w) \in E$. Therefore, without loss of generality, $v \in S_w$ and the construction (2.47) implies

$$\begin{aligned}
\mathbb{P}(Y_v = Y_w = 0) = \mathbb{P}(X_v = 0, X_{S_v} = \vec{1}, X_w = 0, X_{S_w} = \vec{1}) \\
\le \mathbb{P}(X_v = 1, X_v = 0) = 0\,.
\end{aligned}$$

Thus (2.8c) holds. Let $U, W \subsetneq \mathbb{N}$ with $d(U,W) > 1$. The construction (2.47) implies, that Y_U is a function of $X_{U \cup S_U}$, with $S_U := \biguplus_{u \in U} S_u$, and Y_W a function of an analogous $X_{W \cup S_W}$. Condition (2.48) asserts that $S_U \cap S_W = \emptyset$, whence $(U \cup S_U) \cap (W \cup S_W) = \emptyset$. Therefore Y_U is independent of Y_W and (2.8a) holds. □

2.6.2 Analytic warm up

This section analyses some helper expressions helpful in the following sections. Recall that for us $0^0 = 1$.

Proposition 52. *Fix $k, l \ge 1$ and let*

$$h_{k,l}: \quad [0,1] \to \mathbb{R} \qquad z \mapsto z^k (1-z)^l\,. \tag{2.49}$$

$h_{k,l}$ attains its maximum at $\frac{k}{k+l}$ with value $\frac{k^k l^l}{(k+l)^{(k+l)}}$. Furthermore it is strictly monotone growing in $[0, \frac{k}{k+l}]$ and strictly monotone falling in $[\frac{k}{k+l}, 1]$.

Proof. Everything follows from $\frac{\partial h_{k,l}}{\partial z}(z) = z^{k-1}(1-z)^{l-1}[k - (k+l)z]$. □

Remark. We will often need a solution of $q = h_{k,l}(z)$. In the common case where there are two solutions we will always choose the one closer to 1.

Remark. It has already been noted in [She85] that

$$\lim_{k\to\infty} k\frac{k^k}{(k+1)^{(k+1)}} = \lim_{k\to\infty}\left(1-\frac{1}{k+1}\right)^{k+1} = \frac{1}{e}\,. \tag{2.50}$$

Proposition 53. *For $D \geq 2$, we have*

$$\frac{(D-1)^{(D-1)}}{D^D} = \max_{z\in[1,D-1]}\left\{\frac{(z-1)^{(z-1)}}{z^z}\frac{(D-z-1)^{(D-z-1)}}{(D-z)^{(D-z)}}\right\} \tag{2.51a}$$

$$\frac{(D-1)^{(D-1)}}{D^D} = \max_{z\in[1,D-1]}\left\{\frac{z^z(D-z)^{(D-z)}}{D^D}\right\} \tag{2.51b}$$

$$\frac{(D-1)^{(D-1)}}{D^D} = \min_{z\in[1,D-1]}\left\{\frac{z^z}{(z+1)^{(z+1)}}\right\}. \tag{2.51c}$$

Proof. (2.51c): Regard

$$\begin{aligned} F(z) &:= \frac{z^z}{z+1)^{z+1)}}\,,\\ L(z) &:= \log F(z) = z\log z - (z+1)\log(z+1)\,,\\ \frac{\partial L}{\partial z}(z) &= \log z - \log(z+1) < 0\,. \end{aligned}$$

This shows the convexity of L and F, by exponentiating.

(2.51b): Let

$$\begin{aligned} F(z) &:= \frac{z^z(D-z)^{(D-z)}}{D^D}\,,\\ L(z) &:= \log F(z) = z\log z + (D-z)\log(D-z) - D\log D\,,\\ \frac{\partial L}{\partial z}(z) &= \log z - \log(D-z)\,,\\ \frac{\partial^2 L}{\partial z^2}(z) &= \frac{1}{z} + \frac{1}{D-z} > 0\,. \end{aligned}$$

Thus L is convex with a global minimum at $z = \frac{D}{2}$ and global maxima at $z \in \{1, D-1\}$. Conclude by taking exponentials.

(2.51a): Let

$$\begin{aligned} F(z) &:= \frac{(z-1)^{(z-1)}}{z^z}\frac{(D-z-1)^{(D-z-1)}}{(D-z)^{(D-z)}}\,,\\ L(z) &:= \log F(z)\,,\\ \frac{\partial L}{\partial z}(z) &= \log(z-1) - \log z - \log(D-z-1) + \log(D-z)\,,\\ \frac{\partial^2 L}{\partial z^2}(z) &= \frac{1}{z-1} - \frac{1}{z} + \frac{1}{D-z-1} - \frac{1}{D-z} > 0\,. \end{aligned}$$

Hence L is a convex function with a global minimum at $z = \frac{D}{2}$ and global maxima at $z \in \{1, D-1\}$. Conclude by exponentiating.

□

2.6.3 On the k-fuzz of $\mathbb{Z}$

In this section we deal with Shearer's measure on $\mathbb{Z}_{(k)}$, the *k-fuzz* of $\mathbb{Z}$. It is the graph with vertices $\mathbb{Z}$ and edges for every pair of integers at distance less than or equal to k. Recall that an X-valued process indexed by $\mathbb{Z}$ is called a *$(k+1)$-factor* iff there exists a measurable function $f : [0,1]^{(k+1)} \to X$, such that for every $n \in \mathbb{Z} : X_n = f(U_n, \ldots, U_{n+k})$, where $\{U_n\}_{n\in\mathbb{Z}}$ is a i.i.d. sequence of Uniform$([0,1])$ rvs. It follows that every $(k+1)$-factor is k-independent, stationary and has $\mathbb{Z}_{(k)}$ as strong dependency graph. This section is a rewrite of [MT12, section 4.2].

We derive the critical value $p_{sh}^{\mathbb{Z}_{(k)}}$ in proposition 54 (thus validating (3.12)), construct $\mu_{\mathbb{Z}_{(k)},p}$ explicitly in model 55 as a $(k+1)$-factor and derive asymptotic properties in proposition 56. For $k \in \mathbb{N}_0$ and $p \in [p_{sh}^{\mathbb{Z}_{(k)}}, 1]$, let $\xi := \xi(p,k)$ be the *unique solution* of

$$h_{k,1}(\xi) = q \tag{2.52}$$

lying in the interval $[k/(k+1), 1]$ (see proposition 52). Denote by $[N]_{(k)}$ the k-fuzz of a line of N points and by $\mathbb{N}_{(k)}$ the k-fuzz of $\mathbb{N}$. It is easy to see that $p_{sh}^{\mathbb{N}_{(k)}} = p_{sh}^{\mathbb{Z}_{(k)}}$ and $\mu_{\mathbb{N}_{(k)},p}$ is just the projection of $\mu_{\mathbb{Z}_{(k)},p}$. Hence all the properties of and estimates for $\mu_{\mathbb{Z}_{(k)},p}$ stated in the following also hold for $\mu_{\mathbb{N}_{(k)},p}$.

Proposition 54.

$$p_{sh}^{[N]_{(k)}} \xrightarrow[N\to\infty]{} 1 - \frac{k^k}{(k+1)^{(k+1)}} = p_{sh}^{\mathbb{Z}_{(k)}} = p_{sh}^{\mathbb{N}_{(k)}} \,. \tag{2.53}$$

An explicit construction of Shearer's measure on $\mathbb{Z}_{(k)}$ is given by:

Model 55. Let $p \geq p_{sh}^{\mathbb{Z}_{(k)}}$ and $X := \{X_n\}_{n\in\mathbb{Z}}$ be i.i.d. Bernoulli rvs with parameter ξ as in (2.52). Define $Z := \{Z_n\}_{n\in\mathbb{Z}}$ by

$$\forall\, n \in \mathbb{Z} : \quad Z_n := 1 - (1 - X_n) \prod_{i=1}^{k} X_{n-i} \,, \tag{2.54}$$

then Z is $\mu_{\mathbb{Z}_{(k)},p}$-distributed.

If $k = 0$, then the empty product in (2.54) disappears and $Z = X$, that is $\mu_{\mathbb{Z}_{(0)},p}$ is a Bernoulli product measure with parameter p. Accordingly $p_{sh}^{\mathbb{Z}_{(0)}} = 0$.

A result of Aaronson, Gilat, Keane & de Valk [AGKdV89, result 4(i) on page 140] on the question of the representation of certain stationary 1-independent $\{0,1\}$-valued processes on $\mathbb{Z}$ as 2-factors implies that $\mu_{[n]_1,p}$ is not representable as a 2-factor, for $p < \frac{3}{4}$ and $n \in \mathbb{N}$. This statement is easily extended to assert non-representation of $\mu_{[n]_{(k)},p}$ as a $(k+1)$-factor for every $k, n \in \mathbb{N}$ and $p < p_{sh}^{\mathbb{Z}_{(k)}}$. The core of their proof is a recursive construction of such a process by a dynamic system. If all the trajectories of the system stay positive, then they the trajectories encode the process' distribution. In the present case the sequence $(\beta_n)_{n\in\mathbb{N}}$ in the proof of proposition 54 does not remain positive, which entails that some trajectories of the dynamic system become negative. See also section 6.4.3, for a related argument on grid-like graphs.

On the other hand, if one fixes N and $p \in [p_{sh}^{[N]_{(k)}}, p_{sh}^{\mathbb{Z}_{(k)}}[$, one can get something close to a factor representation. Let $(X_n)_{n=1}^N$ be a collection of independent rvs, with X_n Bernoulli(β_n)-distributed. Then the same rule as in (2.54), truncated for the first k indices, yields a $\mu_{[N]_{(k)},p}$-distributed BRF $(Z_n)_{n=1}^N$. This is nothing else than the intrinsic construction described in section 2.6.1.

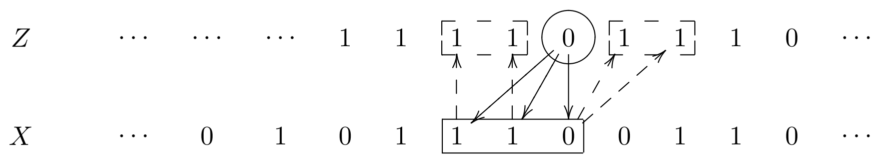

Figure 2.1: A partial view of Shearer's measure on the 2-fuzz of $\mathbb{Z}$. The lower row shows a realization of X, the upper row the resulting one of Z. We point out a 0 in Z, the realizations on its underlying nodes in X (solid downward arrows) and the effect of the zero-one switch (dashed upward arrows), resulting in 1s on its neighbours up to distance 2.

Proof of proposition 54. The inequality $p_{sh}^{\mathbb{Z}_{(k)}} \leq 1 - \frac{k^k}{(k+1)^{(k+1)}}$ follows from model 55. The second inequality $p_{sh}^{\mathbb{Z}_{(k)}} \geq 1 - \frac{k^k}{(k+1)^{(k+1)}}$ follows from [LSS97, theorem 2.1]. We repeat the argument for completeness. For $p \geq p_{sh}^{\mathbb{Z}_{(k)}}$ let $\beta_n := \alpha_{[n-1]}^n(p)$. The sequence $(\beta_n)_{n\in\mathbb{N}}$ is decreasing by (2.26a) and translation-invariance of $\mu_{\mathbb{Z}_{(k)},p}$. It has limit $\beta \in [0,1]$. Taking the limit of the fundamental identity (2.37) results in the following, well-defined equation:

$$\beta = 1 - \frac{q}{\beta^k}.$$

But by proposition 52 this equation, rewritten as $q = (1-\beta)\beta^k = h_{k,1}(\beta)$, has only solutions for $q \leq \frac{k^k}{(k+1)^{(k+1)}}$. Hence $1 - \frac{k^k}{(k+1)^{(k+1)}} \leq p_{sh}^{\mathbb{Z}_{(k)}}$. □

Proof of model 55. By construction $\mathbb{P}(Z_n = 0) = \xi^k(1-\xi) = h_{k,1}(\xi) = q$ and

$$\begin{aligned}\mathbb{P}(Z_n = 0) &= \mathbb{P}(X_{n-1} = \ldots = X_{n-k} = 1, X_n = 0)\\ &= \mathbb{P}(Z_{n-k} = \ldots = Z_{n-1} = 1, Z_n = 0, Z_{n+1} = \ldots = Z_{n+k} = 1).\end{aligned}$$

This *zero-one switch* (see figure 2.1) guarantees that vertices with distance less than or equal to k can never index a 0 in the same realization. Therefore Z has no realizations containing neighbouring 0s with respect to $\mathbb{Z}_{(k)}$ as well as the right dependency graph and marginals. Using the characterization (2.8) we see that Z is $\mu_{\mathbb{Z}_{(k)},p}$-distributed. □

For $k \in \mathbb{N}_0$ fixed define the strictly monotone decreasing function

$$f_k: \quad \{0,\ldots,k\} \to \mathbb{R} \qquad g \mapsto \begin{cases} \frac{(g+1)\xi - g}{g\xi - (g-1)} & \text{if } k \geq 1, \\ \xi & \text{if } k = 0. \end{cases} \tag{2.55}$$

Proposition 56. *We have for every $k \in \mathbb{N}_0$ and $p \geq p_{sh}^{\mathbb{Z}_{(k)}}$ the minoration*

$$\forall \text{ finite } B \subseteq \mathbb{Z} \setminus \{0\} : \qquad \mu_{\mathbb{Z}_{(k)},p}(Y_0 = 1 | Y_B = \vec{1}) \geq f_k(g_B)\,, \tag{2.56a}$$

where $g_B := 0 \vee (k+1-d_B)$ and $d_B := \min\{|n| : n \in B\}$. In particular we have

$$\forall n \in \mathbb{N} : \quad \mu_{\mathbb{Z}_{(k)},p}(Y_n = 1 | Y_{[n-1]} = \vec{1}) \geq \xi\,, \tag{2.56b}$$

and the strict bounds

$$\forall \varepsilon > 0 : \exists C > 0, \exists N \in \mathbb{N} : \forall n \geq N : \qquad \xi^n \leq \Xi_{[n]_{(k)}}(p) \leq C[(1+\varepsilon)\xi]^n\,. \tag{2.56c}$$

This yields the limit

$$\lim_{n\to\infty} \frac{\log \Xi_{[n]_{(k)}}(p)}{n} = \lim_{n\to\infty} \log \Xi_{[n]_{(k)}}(p) = \log \xi\,. \tag{2.56d}$$

Remark. The minimality of Shearer's measure (2.28a) implies that these lower bounds also hold for every k-independent BRF on $\mathbb{Z}$ and $\mathbb{N}$ with marginal parameter $p \geq p_{sh}^{\mathbb{Z}_{(k)}}$ respectively.

Proof. Fix $p \geq p_{sh}^{\mathbb{Z}_{(k)}}$. In the proof of proposition 54 we see that $(\beta_n)_{n\in\mathbb{N}}$ is a strictly monotone falling sequence with $\beta_n \xrightarrow[n\to\infty]{} \beta \geq \frac{k}{k+1}$. As β fulfils $q = h_{k,1}(\beta)$ we have $\beta = \xi$. Hence $\beta_n \geq \xi$, yielding (2.56b). The monotonicity of $(\beta_n)_{n\in\mathbb{N}}$ implies that

$$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : \qquad \beta_n \leq (1+\varepsilon)\beta = (1+\varepsilon)\xi\,.$$

Hence for $n \geq N$ we have

$$\Xi_{[n]_{(k)}}(p) = \prod_{i=1}^{n} \beta_i \leq \prod_{i=N+1}^{n} \beta_i \leq (1+\varepsilon)^{n-N}\xi^{n-N} = \frac{1}{[(1+\varepsilon)\xi]^N}[(1+\varepsilon)\xi]^n\,.$$

This proves (2.56c) upon setting $C(\varepsilon) := [(1+\varepsilon)\xi]^{-N}$. The limit (2.56d) follows directly from (2.56c).

For (2.56a) we differentiate according to the shape of B. If $d_B > k$, then k-independence implies that $\mu_{\mathbb{Z}_{(k)},p}(Y_0 = 1|Y_B = \vec{1}) = p \geq \xi = f_k(0)$.

If $d_B \leq k$ let $B_\pm := B \cap \mathbb{Z}_\pm$ and $d_{B_\pm} := \inf\{|n| : n \in B_\pm\}$. Thus $d_B = d_{B_-} \wedge d_{B_+}$. In the first case $d_{B_-} > k$ and $d_{B_+} \leq k$. Let $\{b_1, \ldots, b_m\} := B_+ \cap [k]$ with $b_1 < \ldots < b_m$. Hence

$$\begin{aligned}
&1 - \mu_{\mathbb{Z}_{(k)},p}(Y_0 = 1|Y_B = \vec{1}) \\
&= 1 - \mu_{\mathbb{Z}_{(k)},p}(Y_0 = 1|Y_{B_+} = \vec{1}) && \text{by } k\text{-independence} \\
&= \frac{q}{\prod_{i=1}^m \mu_{\mathbb{Z}_{(k)},p}(Y_{b_i} = 1|Y_{B_+\setminus\{b_1,\ldots,b_{i-1}\}} = \vec{1})} && \text{fundamental identity (2.11)} \\
&\leq \frac{q}{\xi^m} && \text{by induction over } |B_+| \\
&\leq (1-\xi)\xi^{k-m} && \text{as } q = (1-\xi)\xi^k
\end{aligned}$$

$$\leq 1-\xi \qquad \text{as } m \leq k\,.$$

This also holds in the symmetric case with $d_{B_-} \leq k$ and $d_{B_+} > k$.

The final case is $d_{B_+} \leq k$ and $d_{B_-} \leq k$. Assume without loss of generality that $d_B = d_{B_-} \leq d_{B_+}$ and let $\{a_n, \ldots, a_1\} := B_- \cap \{-k, \ldots, -1\}$ with $a_n < \ldots < a_1$. Applying the fundamental identity (2.11) and induction over $|B|$ we get

$$\begin{aligned}
&1 - \mu_{\mathbb{Z}_{(k)},p}(Y_0 = 1 | Y_B = \vec{1})\\
&= \frac{q}{\prod_{j=1}^{n} \mu_{\mathbb{Z}_{(k)},p}(Y_{a_j} = 1 | Y_{B_- \setminus \{a_1,\ldots,a_{j-1}\}} = \vec{1}, Y_{B_+} = \vec{1})}\\
&\quad \times \frac{1}{\prod_{i=1}^{m} \mu_{\mathbb{Z}_{(k)},p}(Y_{b_i} = 1 | Y_{B_- \setminus \{a_1,\ldots,a_n\}} = \vec{1}, Y_{B_+ \setminus \{b_1,\ldots,b_{i-1}\}} = \vec{1})}\\
&\leq \frac{q}{\prod_{j=1}^{n} f_k(k+a_j) \prod_{i=1}^{m} f_k(0)}\\
&\leq \frac{q}{\prod_{j=d_B}^{k} f_k(k-j) \prod_{i=1}^{k} f_k(0)}\\
&= \frac{(1-\xi)\xi^k}{\left[\prod_{j=d_B}^{k} \frac{(k+1-j)\xi-(k-j)}{(k-j)\xi-(k-1-j)}\right] \xi^k}\\
&= \frac{1-\xi}{(k+1-d_B)\xi - (k-d_B)}\,.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mu_{\mathbb{Z}_{(k)},p}(Y_0 = 1 | Y_B = \vec{1}) &\geq 1 - \frac{1-\xi}{(k+1-d_B)\xi - (k-d_B)}\\
&= \frac{(k+2-d_B)\xi - (k+1-d_B)}{(k+1-d_B)\xi - (k-d_B)} = f_k(k+1-d_B)\,.
\end{aligned}$$

□

2.6.4 On the regular tree

The classic example is the infinite d–regular tree in the homogeneous case. It was first solved implicitly by [She85, pp 243-244 before theorem 2], where he gave one half of the proof of proposition 57.

Proposition 57. *Let* $\mathbb{T}_d = (V, E)$ *be the* infinite d–regular tree. *Then*

$$p_{sh}^{\mathbb{T}_d} = 1 - \frac{(D-1)^{(D-1)}}{D^D}\,. \tag{2.57}$$

The result (2.57) together with (2.50) shows, that the homogeneous escaping LLL in theorem 40 is optimal. In particular, we can construct Shearer's measure explicitly:

Model 58. Let $p \geq 1 - \frac{(D-1)^{(D-1)}}{D^D}$. Let ξ be the bigger solution of

$$q = h_{d-1}(\xi) = (1-\xi)\xi^{d-1}\,. \tag{2.58}$$

Orient $\mathbb{T}_d$ towards one end of its boundary and call s_v the bigger neighbour of the vertex v (as in a *horocyclic tree* [CKW94]). Let $X := (X_v)_{v\in V}$ be Π^{ξ}_V-distributed. Define the BRF $Y := (Y_v)_{v\in V}$ by

$$\forall\, v \in V: \quad Y_v := 1 - (1 - X_v) \prod_{s_v \neq w \in \mathcal{N}(v)} X_w\,. \tag{2.59}$$

Then Y is $\mu_{\mathbb{T}_d,p}$-distributed.

The construction in model 58 allows a detailed control of the various quantities of interest of $\mu_{\mathbb{T}_d,p}$:

Proposition 59. *Let $T_n := (V_n, E_n)$ be a full rooted tree of rank $d-1$, depth n and with root o. Let $p \geq p^{\mathbb{T}_d}_{sh}$ and ξ be the bigger solution of $q = h_{d-1}(\xi)$. Let*

$$m_n := |V_n| = \begin{cases} \frac{(d-1)^{n+1}-1}{d-2} & \text{if } d \geq 3 \\ n+1 & \text{if } d = 2\,. \end{cases} \tag{2.60a}$$

Then

$$\forall\, n \in \mathbb{N}: \quad \alpha^o_{V_n\setminus\{o\}}(p) \geq \xi\,. \tag{2.60b}$$

and

$$\forall\, n \in \mathbb{N}, \varepsilon > 0 : \exists\, C > 0: \quad \xi^{m_n} \leq \Xi_{T_n}(p) \leq C[(1+\varepsilon)\xi]^{m_n} \tag{2.60c}$$

Therefore

$$\lim_{n\to\infty} \frac{\log \Xi_{T_n}(p)}{m_n} = \lim_{n\to\infty} \log \alpha^o_{V_n\setminus\{o\}}(p) = \log \xi\,. \tag{2.60d}$$

Proof of model 58. The construction (2.59) fulfils the conditions (2.48). Thus proposition 51 asserts that Y fulfils the characterization (2.8). □

Proof of proposition 57. One inequality is given by model 58. For the other inequality let T be the infinite, rank $(d-1)$ tree with root o. Suppose that $p \geq p^{\mathbb{T}_d}_{sh}$ and let $a := \alpha^o_{V\setminus\{o\}}(p)$. Using the limit fundamental identity (2.37) and the fact that T contains several disjoint copies of itself we arrive at $a = 1 - p/a^{d-1}$. Rewrite this expression to $1 - p = (1-a)a^{d-1}$ and see, that non-negative solutions are only possible for $p \geq 1 - \frac{(D-1)^{(D-1)}}{D^D}$. □

Proof of proposition 59. The minoration (2.60b) follows directly from the construction (2.59). The bounds (2.60c) are a consequence of factorizing $\Xi_{T_n}(p)$ as a product of escaping OVOEPs and their monotonicity properties. The limit (2.60d) is a direct consequence of this. □

2.6.5 A quasi-transitive example

Let G be the Cayley graph of the group $\langle a, b | a^2, b^3\rangle$ with respect to the generators a and b. We give the value of p^G_{sh} and give two constructions of a $\mu_{G,p}$-distributed BRF, valid for the whole range $[p^G_{sh}, 1]$. This example intends to present a more elaborate example, which is still tractable, and serves as a motivation for conjecture 63.

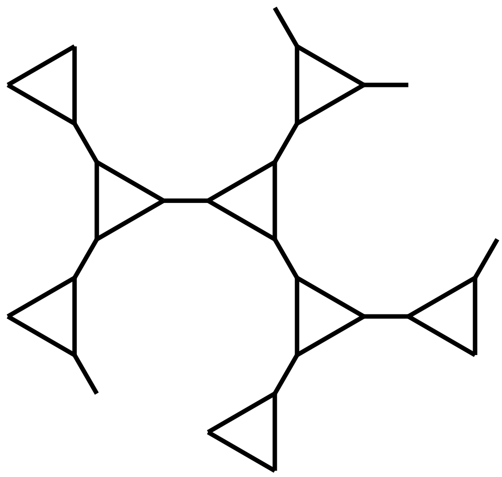

Figure 2.2: A small portion of the Cayley graph of $\langle a, b | a^2, b^3 \rangle$.

Proposition 60. *We have*

$$p_{sh}^G = 2(\sqrt{2} - 1)\,. \tag{2.61}$$

Picture the graph G as either a regular tree $\mathbb{T}_3$ with every vertex replaced by a triangle or as a 3-regular tree of triangles (see figure 2.2). We want to orient G. Let R be a bi-infinite, transitive, geodesic ray R in G. We choose a base vertex $o \in R$ and identify the vertices of R with $\mathbb{Z}_{(1)}$, by identifying o with 0. We think of the end of R going towards ∞ as *up* and picture G as hanging down from this end. Define the *height function* $h : V \to \mathbb{Z}$ on R by this identification and extend it to $V \setminus R$ by

$$h : \quad V \to \mathbb{Z} \qquad v \mapsto \min \{d(v, w) + h(w) : \quad w \in R\}\,. \tag{2.62}$$

It follows, that h takes exactly two values with difference 1 on every triangle T of G, with the bigger value only once. Each vertex $v \in V$ has three neighbours: two forming a triangle with it and one in the next triangle. Call $\mathfrak{p}(v)$ the unique neighbour of v with $h(\mathfrak{p}(v)) = h(v)+1$. If $h(v)$ is the bigger value in its triangle, then its parent is the outside node. Call the set of these vertices $\mathcal{U}$. If $h(v)$ is one of the two smaller values in its triangle, then $\mathfrak{p}(v)$ is the $\mathcal{A}$ vertex of its triangle. By making an arbitrary choice at each triangle T, we split these lower vertices of T into the sets $\mathcal{L}_+$ and $\mathcal{L}_-$. Each vertex v in $\mathcal{L}_+$ has a unique sibling in $\mathcal{L}_-$ and vice-versa. Summing up, the neighbours of $v \in V$ are

$$\begin{aligned} v \in \mathcal{U} \;&\Rightarrow\; \mathfrak{p}(v) \in \mathcal{L}_\pm, c_+(v) \in \mathcal{L}_+, c_-(v) \in \mathcal{L}_- \\ v \in \mathcal{L}_\pm \;&\Rightarrow\; \mathfrak{p}(v) \in \mathcal{U}, s_\mp(v) \in \mathcal{L}_\mp, c(v) \in \mathcal{U}\,. \end{aligned} \tag{2.63}$$

Model 61. For $q \leq 3 - 2\sqrt{2}$, let u and $l_\pm$ be solutions of

$$q = (1-u)l_+l_- = (1-l_+)ul_- = (1-l_-)u\,. \tag{2.64}$$

Let $X^{\mathcal{U}} := (X_v^{\mathcal{U}})_{v\in\mathcal{U}}$ be $\Pi_u^{\mathcal{U}}$-distributed, $X^{\mathcal{L}_\pm} := (Z_v^{\mathcal{L}_\pm})_{v\in\mathcal{L}_\pm}$ be $\Pi_{l_\pm}^{\mathcal{L}_\pm}$-distributed, all three BPFs independently of each other. Define the BRF $Y := (Y_v)_{v\in V}$ by

$$\forall\, v \in V: \quad Y_v := \begin{cases} 1-(1-X_v^{\mathcal{U}})X_{c_+(v)}^{\mathcal{L}_+}X_{c_-(v)}^{\mathcal{L}_-} & \text{if } v \in \mathcal{U} \\ 1-(1-X_v^{\mathcal{L}_+})X_{c(v)}^{\mathcal{U}}X_{s_-(v)}^{\mathcal{L}_-} & \text{if } v \in \mathcal{L}_+ \\ 1-(1-X_v^{\mathcal{L}_-})X_{c(v)}^{\mathcal{U}} & \text{if } v \in \mathcal{L}_-\,. \end{cases} \tag{2.65}$$

Then Y is $\mu_{G,p}$-distributed and called the *zero-one switch model*.

Let $\mathcal{T}$ be the set of all *triangles* in G and $\mathcal{B}$ the set of all *brigdes* in G, that is all edges not part of a triangle. Each triangle $T \in \mathcal{T}$ has vertices $u(T)$,$l_+(T)$ and $l_-(T)$. Each brigde $b \in \mathcal{B}$ has endpoints $u(b)$ and $l(b)$. For a vertex $v \in V$ let $T(v)$ and $b(v)$ be its incident triangle and bridge respectively.

Model 62. Let $X := (X_T)_{T\in\mathcal{T}}$ and $Z := (Z_b)_{b\in\mathcal{B}}$ be two independent product fields, which are $\{\uparrow,\searrow,\swarrow\}$-valued with $\mathbb{P}(X_T=\searrow) = \mathbb{P}(X_T=\swarrow) = \frac{\sqrt{2}-1}{\sqrt{2}}$ and $\mathbb{P}(X_T=\uparrow) = \sqrt{2}-1$ and $\{\uparrow,\downarrow\}$ with $\mathbb{P}(Z_b=\uparrow) = 2-\sqrt{2}$ and $\mathbb{P}(Z_b=\downarrow) = \sqrt{2}-1$ respectively. Define the BRF $Y := (Y_v)_{v\in V}$ by

$$\begin{aligned} &\forall\, v \in V: \quad 1 - Y_v \\ &:= \begin{cases} [X_{T(v)}=\uparrow][Z_{b(v)}=\downarrow] & \text{if } v \in \mathcal{U} \Leftrightarrow u(T(v)) = v = l(b(v)) \\ [X_{T(v)}=\swarrow][Z_{b(v)}=\uparrow] & \text{if } v \in \mathcal{L}_+ \Leftrightarrow l_+(T(v)) = v = u(b(v)) \\ [X_{T(v)}=\searrow][Z_{b(v)}=\uparrow] & \text{if } v \in \mathcal{L}_- \Leftrightarrow l_-(T(v)) = v = u(b(v))\,. \end{cases} \end{aligned} \tag{2.66}$$

Then Y is $\mu_{G,2(\sqrt{2}-1)}$-distributed and called the *arrow model*.

Proof of model 61. The construction (2.59) fulfils the conditions (2.48). Thus proposition 51 asserts that Y fulfils the characterization (2.8). □

Proof of model 62. The construction from the arrows make it evident, that G is a strong dependency graph of Y. To have $Y_v = 0$ both arrows of $X_{T(v)}$ and $Z_{b(v)}$ have to point to v, thus excluding the possibility of a 0 at either of the three neighbours of v, as it shares either $T(v)$ or $b(v)$ with them. The marginal parameter of Y is

$$\mathbb{P}(Y_v = 0) = \begin{cases} \mathbb{P}(X_{T(v)}=\uparrow, Z_{b(v)}=\downarrow) = (\sqrt{2}-1)^2 = 3-2\sqrt{2} & \text{if } v \in \mathcal{U} \\ \mathbb{P}(X_{T(v)}=\swarrow, Z_{b(v)}=\uparrow) = (2-\sqrt{2})\frac{\sqrt{2}-1}{\sqrt{2}} = 3-2\sqrt{2} & \text{if } v \in \mathcal{L}_+ \\ \mathbb{P}(X_{T(v)}=\searrow, Z_{b(v)}=\uparrow) = (2-\sqrt{2})\frac{\sqrt{2}-1}{\sqrt{2}} = 3-2\sqrt{2} & \text{if } v \in \mathcal{L}_-\,. \end{cases} \tag{2.67}$$

Thus Y fulfils the characterization (2.8) and is $\mu_{G,2(\sqrt{2}-1)}$-distributed. □

Proof of proposition 60. The inequality $p_{sh}^G \geq 2(\sqrt{2}-1)$ follows from either one of model 61 or 62.

Let $p \geq p_{sh}^G$. Choose $u \in \mathcal{U}$ and $v \in \mathcal{L}_+$. For $n \in \mathbb{N}_0$ define the subgraphs A_n B_n and C_n (see figure 2.3) by

$$\begin{aligned} A_n &:= G(\{w \in V : \quad h(v) - n \leq h(w) \leq h(v), d(u,w) \leq d(\mathfrak{p}(u), w)\}) \\ B_n &:= G(\{w \in V : \quad h(v) - n \leq h(w) \leq h(v), d(v,w) \leq d(\mathfrak{p}(v), w)\}) \\ C_n &:= G(\{w \in V : h(v) - n \leq h(w) \leq h(v), d(v,w) \leq d(\mathfrak{p}(v), w) \wedge d(s_- v, w)\}) \,. \end{aligned}$$

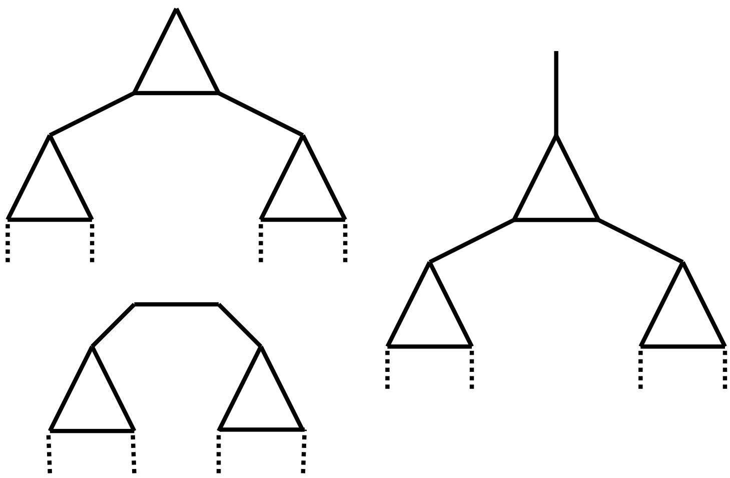

Figure 2.3: The uppermost part of the subgraphs induced by the sets A_n (top left), B_n (bottom left) and C_n (right).

We abuse notation and also let $A_n := \Xi_{G_{n,1}}(p)$,$B_n := \Xi_{G_{n,2}}(p)$ and $C_n := \Xi_{G_{n,3}}(p)$. The fundamental identity (2.11) yields

$$A_n = B_{n-1} - q\, A_{n-2}^2 \tag{2.68a}$$
$$B_n = A_{n-1} C_n - q\, B_{n-2} A_{n-1} = A_{n-1}(C_n - q\, B_{n-2}) \tag{2.68b}$$
$$C_n = A_{n-1} - q\, B_{n-2} \,. \tag{2.68c}$$

We can reduce this too

$$\begin{aligned} A_{n+1} + q\, A_{n-1}^2 &= B_n \\ &= A_{n-1}(C_{n-1} - qB_{n-2}) \\ &= A_{n-1}(A_{n-1} - 2qB_{n-2}) \\ &= A_{n-1}(A_{n-1} - 2q(A_{n-1} + qA_{n-3}^2) \,. \end{aligned}$$

Shifting indices we have:

$$A_n + (3q-1)A_{n-2}^2 + 2q^2 A_{n-2} A_{n-4}^2 = 0 \,.$$

For $n = 2m$ even and $D_m := A_{2m}$ we get

$$D_m + (3q-1)D_{m-1}^2 + 2q^2 D_{m-1} D_{m-2}^2 = 0\,.$$

Dividing by D_{m-1}^2, let $\delta_m := \frac{D_m}{D_{m-1}^2}$ be the OVOEP. Thus

$$\delta_m + (3q-1) + \frac{2q^2}{\delta_{m-1}} = 0\,.$$

In the limit $\delta_m \to \delta$ we get

$$\delta^2 + (3q-1)\delta + 2q^2 = 0\,.$$

A necessary condition to have a real solution of this equation is that $q \leq 3-2\sqrt{2}$. The δs in the previous lines correspond to MVOEPs (*multi-vertex open extension probabilities*), adding a triangle at a time. □

Remark. An alternate reduction of (2.68) is to introduce the OVOEPs

$$\alpha_n := \frac{A_n}{B_{n-1}} \qquad \beta_n := \frac{B_n}{A_{n-1}C_n} \qquad \gamma_n := \frac{B_n}{A_{n-1}C_n}\,.$$

This lets us reencode the system (2.68) as

$$\alpha_n = 1 - \frac{q}{\beta_{n-1}\gamma_{n-1}} \qquad \beta_n = 1 - \frac{q}{\alpha_{n-1}\gamma_n} \qquad \gamma_n = 1 - \frac{q}{\alpha_{n-1}}\,. \tag{2.69}$$

If $p \geq p_{sh}^G$, then we can take the limit $n \to \infty$ and arrive at

$$q = (1-\alpha)\beta\gamma \qquad q = (1-\beta)\alpha\gamma \qquad q = (1-\gamma)\alpha\,. \tag{2.70}$$

If we assume $p <$, then $\alpha, \beta, \gamma \in]0,1[$, because the OVOEPs are all escaping (2.25b). Note the equivalence to (2.65).

2.6.6 A conjecture for quasi-transitive tree-like graphs

The previous three sections 2.6.3, 2.6.4 and 2.6.5 have shown three examples of transitive tree-like graphs, where the determination of p_{sh}^G has been possible. At the same time the calculations gave the parameters of an intrinsic construction via OVOEPs as in section 2.6.1. More examples could be shown, but the calculations are tedious and we have always found the same pattern.

In all investigated cases we get a finite linear systems of equations with polynomials in q as coefficients. There are theorems [BKW75], that tell us the solutions are entire functions of q. The problem is, that in every case except $\mathbb{Z}_{(1)}$, these entire functions are impossible to calculate explicitly. Note that the definition of ξ in models 55 and 58 and of (u, l_+, l_-) in model 61, just describe this solution, though.

In our examples, changing to OVOEPs gives rise to a higher order system in several variables. In the investigated cases the solutions are just the parameters of the BPF used for the construction. The parameters itself are limits of OVOEPs and can be seen as fixpoints of the higher order systems. For $q \leq q_{sh}^G$, these higher order systems are contractive.

The recursions are possible, because the fundamental identity, after some steps, factorizes over disjoint subgraphs. This is the point at which the tree-like shape of G comes into play.

All of this leads me to formulate conjecture 63. It is written from the point of view of construction via OVOEPs as in section 2.6.1.

Conjecture 63. *Let $G := (V, E)$ be an infinite, quasi-transitive and connected graph with more than one end containing*

$$\text{no infinite, connected subgraph with just one end.} \tag{2.71}$$

Then there exists a finite set I, such that q_{sh}^G is the solution of

$$\begin{aligned} \max \quad & q \\ st \quad & \begin{cases} \forall\, i \in I : \exists\, I_i \subsetneq I : & q = (1 - x_i) \prod_{j \in I_i} x_j \\ \forall\, i \in I : & x_i \in [0,1]\,. \end{cases} \end{aligned} \tag{2.72}$$

For $p \geq p_{sh}^G$ Shearer's measure $\mu_{G,p}$ can be constructed from a quasi-transitive BPF with quasi-transitive parameters $\vec{x}$, where $\vec{x}$ is a solution of the above set of equations corresponding to q.

Remark. This conjecture should be generalizable to the case of quasi-transitive parameters. The set of admissible values and solutions for (2.72) should be $\mathcal{Q}_G^{sh}$ and $\partial\mathcal{Q}_G^{sh}$ respectively.

The vector $\vec{x}$ solving (2.72) may not be unique.

Sketch of proof of conjecture 63. This sketch describes the general outline of the proof that I envision. Some steps (definition of interface, how to recurse) are only described in informal terms – it is exactly at these points, where the main notational difficulties lie.

The sketch has two parts. In the first part we show, that if Shearer's measure exists, then q must be admissible for (2.72). We regard all OVOEPs as infinite and escaping. Escaping by construction and infinite, as we work directly in the limit. The argument then goes as follows:

- Choose a quasi-transitive, geodesic and bi-infinite ray R. Orient the graph G by identifying R with $\mathbb{Z}_{(1)}$ and say that the end of R identified with ∞ is *up*. We introduce a *height function* h, defined on $V(R)$ by the identification with $\mathbb{Z}_{(1)}$ and extend it to the other vertices $V \setminus R$ by

$$h : \quad V \to \mathbb{Z} \quad v \mapsto \min\left\{d(v,w) + h(w) : \quad w \in V(R)\right\}. \tag{2.73}$$

  This is reminiscent of the construction of a *horocyclic tree* [CKW94].

- Cut between levels 0 and 1 and take one of the resulting connected components. Each such component has a finite interface, otherwise we have a contradiction to (2.71). Think of the *interface* as a description of how disjoint isomorphic embeddings of the connected component into itself are connected to form the connected component itself.

- Enumerate the vertices at the interface (i.e. at the former level 0). Start defining OVOEPs and recurse, enumerating new vertices not fitting an already encountered interface. Finish each level before descending and enumerate as to split the condition of the OVOEPs as soon as possible into disconnected parts.
- By (2.71) the recursion factorizes and by quasi-transitivity only already known interfaces should appear, after a finite number of steps.
- Repeating this procedure for each interface yields a finite system of equations, indexed by the interface and size of the down-set below it. Taking the limit and rearranging the equations yields the restrictions of (2.72).
- The maximality of q^G_{sh} (2.38b) justifies taking the max over all such q.

In the second part we show, that given a q admissible for (2.72), we can construct Shearer's measure. The problem (already hinted at in section 2.6.5) for a straightforward reconstruction is the existence of long circles. That is, in all examples so far, the neighbourhood of each vertex in each connected component of its complement has been connected. This is not when circles exists, for example in the Cayley graph of the group $\langle a, b | a^2, b^3 \rangle$ with respect to the generators a and b. Explicit constructions of Shearer's measure seem to involve always some kind of arrows instead of just zero-one switches. □

Chapter 3

K-independent percolation on trees

3.1 Introduction

If we regard percolation on a tree $\mathbb{T}$, then a natural question is which properties of the percolation and the tree determine the percolation behaviour. One is especially interested in bounds which are not particular to a specific model, but are valid for whole classes of models. The class of models we investigate are k-independent (also called k-dependent in the literature) site (bond) percolations with parameter p, i.e. the probability that a single vertex (edge) is open is p and subsets of vertices (edges) are independent if their distance is greater than k. We look for bounds on the parameter p which guarantee either a.s. percolation or a.s. nonpercolation.

Lyons [Lyo90] first treated this question in the case of independent percolation. He defined the branching number $br(\mathbb{T})$ as a measure of the size of $\mathbb{T}$. Then he showed that it is the characteristic determining the critical probability for independent percolation (see theorem 64), that is the parameter threshold at which nonpercolation switches to percolation.

A recent work by Balister & Bollobás [BB06] deals with the class of 1-independent bond percolations (see theorem 65). There are two continuous functions of the branching number which give tight bounds for a.s. percolation and a.s. nonpercolation of each model in this class.

In section 3.3 we present our results: tight bounds for a.s. percolation and a.s. nonpercolation for every k. The bounds are again continuous functions of $br(\mathbb{T})$, parametrized by k. They are the same for bond and site percolation. A core ingredient is a probability measure introduced by Shearer [She85], which has certain nice minimizing properties (see section 2.2.5). We construct it explicitly on the k-fuzz of $\mathbb{Z}$ in section 2.6.3 and show that it is a $(k+1)$-factor. Shearer's measure minorizes the probability of having an open path of k-independent Bernoulli rvs. This property is already exploited implicitly in the work of Balister & Bollobás. We make this argument explicit by using moment method and capacity arguments motivated by Lyons' proof [Lyo90, Lyo92], supplemented with explicit percolation models inspired by Balister & Bollobás' work [BB06].

3.2 Setup and previous results

Let $G := (V, E)$ be a graph. For every subset H of vertices and/or edges of G denote by $V(H)$ the *vertices induced by* H and by $G(H)$ the *subgraph of* G *induced by* H. We have the *geodesic graph distance* d on both vertices and edges, extended naturally to sets of them. Define the equivalence relation $v \leftrightarrow w$ describing *connectedness* on G. We denote by $\mathcal{N}(v)$ the *neighbours* of a vertex v. The *k-fuzz* (or k^{th} *power*) of G is the graph (V, E'), where E' consists of all distinct pairs of vertices with distance less than or equal to k in G.

We primarily work on a *locally finite tree* $\mathbb{T} := (V, E)$. We consider it to be *infinite*, unless explicitly stated otherwise. Between two nodes v and w we have the unique *geodesic path* $P(v, w)$. For the following definitions root $\mathbb{T}$ at the *root* o and visualize the tree spreading out downwards from the root. Define the *level* $l(v) := d(o, v)$ of a node v and let $L(\mathbb{T}, n) := \{v : l(v) = n\}$ be the n^{th} level of $\mathbb{T}$. *Downpaths* and *-rays* are finite and infinite geodesics, which start at some vertex v and go downwards, thereby avoiding all ancestors of v, respectively. Denote the *boundary* of $\mathbb{T}$ by $\partial\mathbb{T}$, which is the set of all ends of $\mathbb{T}$, identified with the set of all downrays of $\mathbb{T}$ starting at o. For all nodes $v \in V \setminus \{o\}$ there is a unique *parent* denoted by $\mathfrak{p}(v)$. The *confluent* $v \curlywedge w$ is the last common ancestor of two distinct nodes v and w. Define a *minimal vertex cutset* Π to be a finite set of vertices containing no ancestors of itself and delineating a connected component containing o. Denote by $\Pi(o)$ the *set of all minimal vertex cutsets* of o. Finally let $\mathbb{T}^v$ be the *induced subtree of* $\mathbb{T}$ *rooted at* v.

Furthermore we abbreviate $\{1, \dots, n\}$ by $[n]$. As a convention we interpret $[0] := \emptyset$, $0^0 := 1$, empty products as 1 and empty sums as 0.

Recall that a *bond* and *site percolation* on a graph $G := (V, E)$ is an rv taking values in $\{0, 1\}^E$ and $\{0, 1\}^V$ respectively. A percolation *percolates* iff it induces an infinite percolation cluster (connected component) in G with nonzero probability.

We investigate percolations on a tree $\mathbb{T} := (V, E)$ and look for properties of the percolation and the tree influencing the percolation behaviour. For $k \in \mathbb{N}_0$ we consider the class of *k-independent site percolations with parameter p on* $\mathbb{T}$, denoted by $\mathcal{C}^k_p(V)$. A site percolation $Z := \{Z_v\}_{v \in V}$ has parameter p iff the probability that a single site is open equals p. For $W \subseteq V$ let $Z_W := \{Z_v\}_{v \in W}$. The site percolation Z is *k-independent* iff

$$\forall\, U, W \subset V : \quad d(U, W) > k \;\Rightarrow\; Z_U \text{ is independent of } Z_W\,, \tag{3.1}$$

that is events on subsets at distance greater than k are independent. Independence is synonymous with 0-independence. The present paper investigates bounds on the parameter p guaranteeing either a.s. percolation or a.s. nonpercolation for the whole class. We define the *critical values*

$$p^k_{max}(V) := \inf\{p \in [0,1] : \forall \mathcal{P} \in \mathcal{C}^k_p(V) : \mathcal{P} \text{ percolates}\} \tag{3.2a}$$
$$p^k_{min}(V) := \inf\{p \in [0,1] : \exists \mathcal{P} \in \mathcal{C}^k_p(V) : \mathcal{P} \text{ percolates}\}\,. \tag{3.2b}$$

Analogously we define the class $\mathcal{C}^k_p(E)$ of k-independent bond percolations with parameter p on $\mathbb{T}$ and critical values $p^k_{max}(E)$ and $p^k_{min}(E)$.

A λ*-flow* on $\mathbb{T}$ is a function $f : V \mapsto \mathbb{R}_+$ such that

$$\forall\, v \in V : \quad 0 \leq f(v) = \sum_{w:\, \mathfrak{p}(w)=v} f(w) \leq \lambda^{-l(v)}\,. \tag{3.3}$$

Lyons [Lyo90] introduced the *branching number* $br(\mathbb{T})$ as a measure of the size of a tree $\mathbb{T}$:

$$\begin{aligned} br(\mathbb{T}) &:= \sup\{\lambda \geq 1 : \exists \text{ nonzero } \lambda\text{-flow on } \mathbb{T}\} \\ &= \sup\{\lambda \geq 1 : \inf_{\Pi\in\Pi(o)} \sum_{v\in\Pi} \lambda^{-l(v)} > 0\}\,. \end{aligned} \tag{3.4}$$

The duality in (3.4) is due to the max-flow min-cut theorem on infinite graphs [FF62]. The branching number $br(\mathbb{T})$ is independent of the choice of o and equals the *exponential of the Hausdorff dimension of the boundary* $\partial\mathbb{T}$ of $\mathbb{T}$ [LP11, section 1.8]. Throughout this paper we assume $br(\mathbb{T})$ to be finite.

The first known result is due to Lyons [Lyo90], where he characterized the *critical value of independent percolation* ($k = 0$) in terms of $br(\mathbb{T})$:

Theorem 64 ([Lyo90, theorem 6.2])**.**

$$p^0_{min}(V) = p^0_{min}(E) = p^0_{max}(V) = p^0_{max}(E) = \frac{1}{br(\mathbb{T})}\,. \tag{3.5}$$

In the independent case the critical values coincide, since for fixed p there is only one percolation. Lyons' proof is based on moment methods and capacity estimates of percolation kernels. In general, a percolation is *quasi-independent* [Lyo92, section 2.4] iff, using the notation from figure 3.2 on page 69 with $u := v \curlywedge w$ the confluent of v and w, we have an $M > 0$ such that for all v and w

$$\mathbb{P}(o \leftrightarrow v, o \leftrightarrow w | o \leftrightarrow u) \leq M \mathbb{P}(o \leftrightarrow v | o \leftrightarrow u) \mathbb{P}(o \leftrightarrow w | o \leftrightarrow u)\,. \tag{3.6}$$

Equivalently this majorizes the *percolation kernel* (3.22c) by

$$\kappa(v, w) := \frac{\mathbb{P}(o \leftrightarrow v, o \leftrightarrow w)}{\mathbb{P}(o \leftrightarrow v)\mathbb{P}(o \leftrightarrow s)} \leq \frac{M}{\mathbb{P}(o \leftrightarrow u)}\,. \tag{3.7}$$

This way Lyons [Lyo92, section 2.4] used the weighted second moment method to get bounds for the probability of the percolation reaching subsets of $\partial\mathbb{T}$ in terms of their *capacity*, extending the independent case in [Lyo90].

In a recent work, Balister & Bollobás [BB06] deal with the class of 1-independent bond percolations:

Theorem 65 ([BB06]).

$$p^1_{min}(E) = \frac{1}{br(\mathbb{T})^2} \tag{3.8a}$$

$$p^1_{max}(E) = \begin{cases} 1 - \frac{br(\mathbb{T})-1}{br(\mathbb{T})^2} & \textit{if } br(\mathbb{T}) \leq 2 \\ \frac{3}{4} & \textit{if } br(\mathbb{T}) \geq 2. \end{cases} \tag{3.8b}$$

Their proof strategy for $p^1_{min}(E)$ is based on the first moment method and a simple explicit model. We generalize it rather straightforwardly to higher k in section 3.4.5. Their proof for $p^1_{max}(E)$ on the other hand combines a so-called canonical model (discussed in section 3.4.4) with several short and elementary inductive proofs (see [Tem08]). For every $p \geq \frac{3}{4}$ this canonical model minimizes the probability to percolate. They implicitly retrace the weighted second moment method, percolation kernel capacity estimates based on λ-flows and the minimizing property (2.28a), (2.56b) of Shearer's measure [She85] on $\mathbb{Z}$. Alas this inductive approach exploits a few particularities of the case $k = 1$, which we have not been able to abstract from.

3.3 Main results

Our principal result in the setting of section 3.2 (see also figure 3.1 on page 63) is:

Theorem 66. *$\forall\, k \in \mathbb{N}_0$:*

$$p^k_{min}(V) = p^k_{min}(E) = \frac{1}{br(\mathbb{T})^{k+1}} \tag{3.9a}$$

$$p^k_{max}(V) = p^k_{max}(E) = \begin{cases} 1 - \frac{br(\mathbb{T})-1}{br(\mathbb{T})^{k+1}} & \textit{if } br(\mathbb{T}) \leq \frac{k+1}{k} \\ 1 - \frac{k^k}{(k+1)^{(k+1)}} & \textit{if } br(\mathbb{T}) \geq \frac{k+1}{k}\,, \end{cases} \tag{3.9b}$$

where we interpret $\frac{1}{0} := \infty$ in the case $k = 0$.

This theorem is a corollary of the more general theorem 67 upon setting $s = k$ and verifying that (un)rooting a percolation does not change its percolation behaviour (see section 3.4.2).

First we narrow down the definition of the percolation classes we work on. Let $\mathcal{C}^{k,s}_{p,o}(V)$ be the class of *rooted site percolations with parameter* p on $\mathbb{T}$ which are *k-independent along downrays from o* and *s-independent elsewhere*, that is among vertices not on the same downray. We define the *rooted critical values* as

$$p^{k,s}_{max}(V) := \inf\left\{p \in [0,1] : \forall\, o \in V : \forall\, \mathcal{P} \in \mathcal{C}^{k,s}_{p,o}(V) : \mathcal{P} \text{ percolates}\right\} \tag{3.10a}$$

$$p^{k,s}_{min}(V) := \inf\left\{p \in [0,1] : \exists\, o \in V : \exists\, \mathcal{P} \in \mathcal{C}^{k,s}_{p,o}(V) : \mathcal{P} \text{ percolates}\right\}. \tag{3.10b}$$

Analogously we define the class $\mathcal{C}^{k,s}_{p,o}(E)$ of k,s-independent, rooted bond percolations with parameter p on $\mathbb{T}$ and the critical values $p^{k,s}_{min}(E)$ and $p^{k,s}_{max}(E)$. Define the function

$$g_k : \quad [1,\infty] \rightarrow]0,1[\qquad y \rightarrow 1 - \frac{y-1}{y^{k+1}} \tag{3.11}$$

and the value

$$p_{sh}^{\mathbb{Z}_{(k)}} := 1 - \frac{k^k}{(k+1)^{(k+1)}} \,. \tag{3.12}$$

We reveal their motivation in proposition 54 and 76 respectively. Our main result determines the critical values (3.10):

Theorem 67. *$\forall\, k, s \in \mathbb{N}_0$:*

$$p_{min}^{k,s}(V) = p_{min}^{k,s}(E) = \frac{1}{br(\mathbb{T})^{k+1}} \tag{3.13a}$$

$$p_{max}^{k,s}(V) = p_{max}^{k,s}(E) = \begin{cases} 1 - \frac{br(\mathbb{T})-1}{br(\mathbb{T})^{k+1}} = g_k(br(\mathbb{T})) & \text{if } br(\mathbb{T}) \leq \frac{k+1}{k} \\ 1 - \frac{k^k}{(k+1)^{(k+1)}} = p_{sh}^{\mathbb{Z}_{(k)}} & \text{if } br(\mathbb{T}) \geq \frac{k+1}{k}\,, \end{cases} \tag{3.13b}$$

where we interpret $\frac{1}{0} := \infty$ in the case $k = 0$.

We give the proof in section 3.4 and a plot of the results (3.13) in figure 3.1.

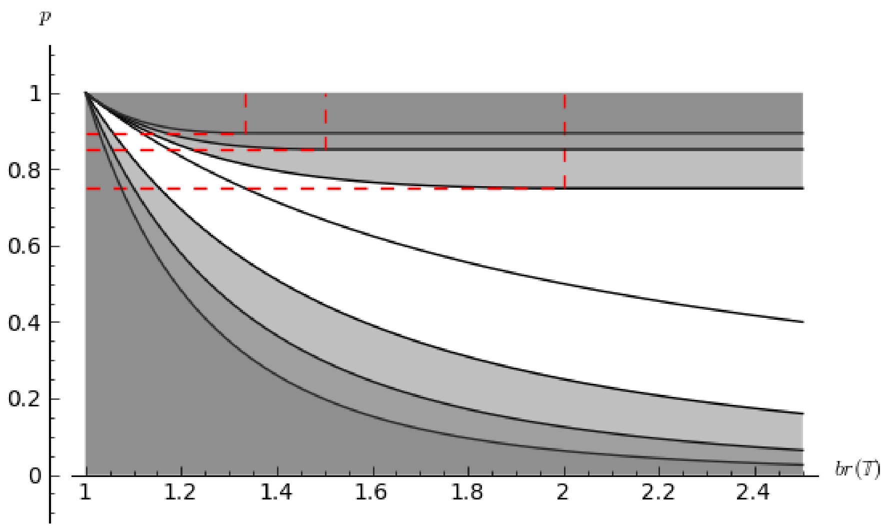

Figure 3.1: (Colour online) The curves of $p_{max}^{k,s}(V)$ and $p_{min}^{k,s}(V)$ for $k \in \{0, 1, 2, 3\}$ and branching numbers in $[1, 2.5]$ delimit increasingly shaded regions. The dashed red lines mark the points $\left(\frac{k+1}{k}, p_{sh}^{\mathbb{Z}_{(k)}}\right)$ for $k \geq 1$, where the behaviour of $p_{max}^{k,s}(V)$ changes.

The critical values (3.13) are *independent of* the root o, the elsewhere-dependence range s and whether we regard bond or site percolation. A change of the root from o to o' turns a k,s-independent percolation at worst into a $(k \vee s),(k \vee s)$-independent percolation. Upon closer inspection one sees that this concerns only elements contained in the ball of radius $d(o, o') + (k \vee s)$ around o'. They are finitely many and one can ignore them as percolation is a tail-event (see the adaption of Kolmogorov's zero-one law in lemma 69), hence the percolation essentially remains k, s-independent. The independence of the parameter s is a consequence of the use of the moment methods, which only take into account the structure of a rooted percolation along downrays. There is a *bijection* from E to $V \setminus \{o\}$ mapping an edge to its endpoint further away from the root o. This implies that $\mathcal{C}^{k,s}_{p,o}(E) \subseteq \mathcal{C}^{k,s+1}_{p,o}(V)$. Furthermore we have $\mathcal{C}^{k,0}_{p,o}(V) = \mathcal{C}^{k,1}_{p,o}(V) = \mathcal{C}^{k,0}_{p,o}(E)$ for $k \geq 1$ and $\mathcal{C}^{0,0}_{p,o}(V) = \mathcal{C}^{0,0}_{p,o}(E)$ as $\mathcal{C}^{0,1}_{p,o}(V) = \emptyset$. This allows the interpretation of our explicit site percolation models (models 81, 83 and 86) as $k, 0$-independent bond percolation models. Hence we *focus exclusively on site percolations* for the remainder of this paper.

We can generalize the single parameter s to a family of finite and unbounded dependency parameters $\vec{s} := \{s_v\}_{v \in V}$. Then the upper bound on $p^{k,\vec{s}}_{max}(V)$ in proposition 76 does not hold anymore. See the counterexample in model 77 and proposition 78. The lower bound on $p^{k,\vec{s}}_{min}(V)$ in propositions 85 and hence the value of $p^{k,\vec{s}}_{min}(V)$ stay valid under these less restrictive conditions and even for $s = \infty$, though.

We determine the critical values by a two-pronged approach. General bounds follow from a direct application of moment arguments [LP11, sections 5.2/5.3] and capacity estimates of percolation kernels [Lyo92, section 1.9]. In section 3.4.6 we show that in every instance where we apply the second moment method our k, s-independent percolations are quasi-independent (3.6). Analysis of a number of explicit percolation models (models 81, 83 and 86) renders the bounds tight. All explicit models are in the class $\mathcal{C}^{k,0}_{p,o}(V)$ and invariant under automorphisms of the rooted tree.

Shearer's measure [She85] on the k-fuzz of $\mathbb{Z}$ (section 2.6.3) minimizes the conditional probability of the event "open for m more steps | open for n steps" along a path of k-independent Bernoulli random variables (see (2.28a)). Our novel contribution is an explicit construction of Shearer's measure on the k-fuzz of $\mathbb{Z}$ (model 55) as a $(k+1)$-factor for $p \geq p^{\mathbb{Z}_{(k)}}_{sh}$ via a zero-one switch ((2.53) and figure 2.1), by reinterpreting calculations from Liggett et al. [LSS97, corollary 2.2]. From the detailed knowledge about Shearer's measure on the k-fuzz of $\mathbb{Z}$ we derive uniform bounds on the percolation kernel over the whole class $\mathcal{C}^{k,s}_{p,o}(V)$, leading to $p^{k,s}_{max}(V)$.

A *back-of-the-envelope derivation of the critical values* (3.10) goes as follows: The simplest infinite rooted tree is a single ray isomorph to $\mathbb{N}$. Let $Z := (Z_n)_{n \in \mathbb{N}}$ be a collection k-independent Bernoulli(p) rvs on $\mathbb{N}$. We have

$$\xi^n \leq \mathbb{P}(Z_{[n]} = \vec{1}) \leq \eta^n \,, \tag{3.14}$$

where the left inequality holds for $p \geq p^{\mathbb{Z}_{(k)}}_{sh}$ with the relation $p = 1 - \xi^k(1-\xi)$, thanks to Shearer (see chapter 2), and the right one always with the relation $\eta^{k+1} = p$, thanks to k-independence. Root $\mathbb{T}$ and suppose that (3.14) carries over to k, s-independent percolation with parameter p. Hence we have a comparison with two independent models with parameters ξ and η, that is

$$\mathbb{P}_\xi(\text{percolates}) \leq \mathbb{P}_p(\text{percolates}) \quad \text{and} \quad \mathbb{P}_p(\text{percolates}) \leq \mathbb{P}_\eta(\text{percolates}) \,, \tag{3.15}$$

where the left inequality holds for $p \geq p_{sh}^{\mathbb{Z}_{(k)}}$. Plugging in $\frac{1}{br(\mathbb{T})}$, the critical value for independent percolation (3.5), for ξ and η we get the g_k part of $p_{max}^{k,s}(V)$ for $br(\mathbb{T}) \leq \frac{k+1}{k}$ and $p_{min}^{k,s}(V)$ for all $br(\mathbb{T})$.

This comparison with two independent models in the last paragraph is solely in terms of the probability to percolate. We have no direct relation between the clusters (like a coupling between the percolations) and in particular no stochastic domination (see section 3.4.7).

Already in the independent case [LP11, section 5] the percolation behaviour at $p = \frac{1}{br(\mathbb{T})}$ depends on additional properties of the tree. This stays the same for $p_{min}^{k,s}(V)$ and the g_k part of $p_{max}^{k,s}(V)$. It is not so for $p = p_{sh}^{\mathbb{Z}_{(k)}}$ and $br(\mathbb{T}) > \frac{k+1}{k}$: here proposition 76 asserts that all $\mathcal{P} \in \mathcal{C}_{p_{sh}^{\mathbb{Z}_{(k)}},o}^{k,s}(V)$ percolate.

Recall that the *diameter* of a percolation cluster is the length of the longest geodesic path contained in it. We call a percolation *diameter bounded* if its percolation cluster diameters are a.s. bounded, i.e.

$$\exists\, D \in \mathbb{N} : \quad \mathbb{P}(\sup\{\operatorname{diam}(C) : C \text{ open cluster in } \mathcal{P}\} \leq D) = 1\,. \tag{3.16}$$

The $p_{sh}^{\mathbb{Z}_{(k)}}$-line admits another interpretation in terms of cluster diameters:

Theorem 68. *For each $\varepsilon > 0$ there exist $p \in]p_{sh}^{\mathbb{Z}_{(k)}} - \varepsilon, p_{sh}^{\mathbb{Z}_{(k)}}[$ and $\mathcal{P} \in \mathcal{C}_{p,o}^{k,0}(V)$ such that $\mathcal{P}$ is diameter bounded. If $p \geq p_{sh}^{\mathbb{Z}_{(k)}}$, then all percolations in $\mathcal{C}_{p,o}^{k,0}(V)$ are not diameter bounded.*

3.4 Proofs

3.4.1 Proof outline of theorems 67 and 68

Proof of theorem 67. We start with some obvious relations between the rooted percolation classes and their critical values, based on the restrictions imposed by k and s. For all $k, k', s, s' \in \mathbb{N}_0$:

$$\text{if } k \leq k' \text{ and } s \leq s' \text{ then } \begin{cases} \mathcal{C}_{p,o}^{k,s}(V) \subseteq \mathcal{C}_{p,o}^{k',s'}(V) \\ p_{max}^{k,s}(V) \leq p_{max}^{k',s'}(V) \\ p_{min}^{k',s'}(V) \leq p_{min}^{k,s}(V) \end{cases} \text{ holds.} \tag{3.17}$$

The first part is the proof for $p_{max}^{k,s}(V)$ in sections 3.4.3 and 3.4.4. To get an upper bound on $p_{max}^{k,s}(V)$ we need to show that every k,s-independent percolation percolates for p close enough to 1. Our approach uses a classical second moment argument, recalled in lemma 73. We relate it to $br(\mathbb{T})$ in proposition 74, with the core ingredient being a sufficient condition for percolation in terms of an exponential bound on the percolation kernel, defined in 3.22c. For k,s-independent percolation proposition 75 reduces this to the problem of bounding the conditional probabilities of extending open geodesic downrays by the right exponential term. Finally proposition 76 uses the minimality of Shearer's measure from theorem 33 and detailed estimates about its structure on $\mathbb{Z}_{(k)}$ in proposition 56 to uniformly guarantee the right exponential term and arrive at (3.25):

$$\forall\, k, s \in \mathbb{N}_0 : \qquad p^{k,s}_{max}(V) \le \begin{cases} g_k(br(\mathbb{T})) & \text{if } br(\mathbb{T}) \le \frac{k+1}{k} \\ p^{\mathbb{Z}_{(k)}}_{sh} & \text{if } br(\mathbb{T}) \ge \frac{k+1}{k}\,. \end{cases}$$

For the lower bound on $p^{k,s}_{max}(V)$ it suffices to exhibit $k, 0$-independent percolation models that do not percolate. We describe two such models, the canonical model 81 and the cutup model 83, both constructed from Shearer's measure. More precisely, in section 3.4.4 we describe a general procedure, called tree-fission, to create a $k, 0$-independent percolation with identical distributions along all downrays from a given k-independent Bernoulli random field over $\mathbb{N}$. When applied to Shearer's measure on $\mathbb{N}_{(k)}$ and a derivative of $[n]_{(k)}$ it yields the canonical model 81 and the cutup model 83 respectively. We then use the first moment method, recalled in lemma 72, to establish their nonpercolation, leading to the following results from (3.30) and (3.31):

$$p^{k,0}_{max}(V) \ge g_k(br(\mathbb{T})) \quad \text{if } br(\mathbb{T}) \in \left[1, \frac{k+1}{k}\right[\qquad \text{and} \qquad p^{k,0}_{max}(V) \ge p^{\mathbb{Z}_{(k)}}_{sh}\,.$$

Conclude by applying the inequality from (3.17).

The second part is the proof for $p^{k,s}_{min}(V)$ in section 3.4.5. Here the argumentation is the reverse of the one for $p^{k,s}_{max}(V)$. To get a lower bound on $p^{k,s}_{min}(V)$ we need to show that every k, s-independent percolation does not percolate for p close enough to 0. We achieve this by a first moment argument in proposition 85, using solely k-independence along downrays. It culminates in (3.32):

$$\forall\, k, s \in \mathbb{N}_0 : \qquad p^{k,s}_{min}(V) \ge \frac{1}{br(\mathbb{T})^{k+1}}\,.$$

For the upper bound on $p^{k,s}_{min}(V)$ we differentiate between $k = 0$ and $k \ge 1$. In the case $k = 0$ we already have a matching upper bound in the upper bound for $p^{0,s}_{max}(V)$ in (3.25). For $k \ge 1$ we describe a percolating $k, 0$-independent percolation model, called the minimal model 86. It is constructed by the tree-fission procedure from section 3.4.4. In proposition 87 we show that it percolates by bounding its percolation kernel with the help of proposition 75 and applying the second moment method adaption from proposition 74, leading to (3.33):

$$\forall\, k \ge 1 : \qquad p^{k,0}_{min}(V) \le \frac{1}{br(\mathbb{T})^{k+1}}\,.$$

Conclude by applying the inequality from (3.17), using the upper bound for $p^{0,s}_{max}(V)$ in the case of $k = 0$. □

Proof of theorem 68. By (2.53) for every $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that $p^{\mathbb{Z}_{(k)}}_{sh} > p^{[N]_{(k)}}_{sh} > p^{\mathbb{Z}_{(k)}}_{sh} - \varepsilon$. Then proposition 84 asserts that the cutup percolation $\mathcal{P}^{cut(k,N)}$ (model 83) is diameter bounded with $D = 4N - 4$.

On the other hand, let $p \ge p^{\mathbb{Z}_{(k)}}_{sh}$ and $Z := \{Z_v\}_{v\in V}$ be in $\mathcal{C}^{k,s}_{p,o}(V)$. We have

$$\forall\, n \in \mathbb{N}, v \in L(\mathbb{T}, n) : \quad \mathbb{P}(Z_{P(o,v)} = \vec{1}) \ge \mu_{\mathbb{Z}_{(k)},p}(Y_{[n]} = \vec{1}) \ge \xi^n > 0\,,$$

where Y is $\mu_{\mathbb{Z}_{(k)},p}$-distributed, we use the minimality of Shearer's measure (2.28b) and the minoration from (2.56b), with $\xi > 0$ from 2.52. This implies that Z is not diameter bounded. □

3.4.2 General tools for percolation on trees

In this section we list some general tools for percolations on trees which allow us to shorten the following proofs. The following *extension* of *Kolmogorov's zero-one law* [Bil95, theorem 36.2] is well known. In particular it encompasses k-independent rvs on a graph G, as they have the k-fuzz of G as their dependency graph.

Lemma 69. *Let $G = (V, E)$ be a locally finite, infinite graph. Let $X := \{X_v\}_{v\in V}$ be a random field with dependency graph G. Then the tail σ-algebra of X is trivial.*

Proof. Let $(V_n)_{n\in\mathbb{N}}$ be an exhausting, strictly monotone growing sequence of finite subsets of V. For $W \subseteq V$ let $\mathcal{A}_W := \sigma(X_W)$ and define the tail σ-algebra $\mathcal{A}_\infty := \bigcap_{n=1}^\infty \mathcal{A}_{V_n^c}$. For an event $B \in \mathcal{A}_\infty$ set $Z_n := \mathbb{E}[\mathbb{I}_B|\mathcal{A}_{V_n}] = \mathbb{I}_B$. Then we have a a.s. constant martingale with $\lim_{n\to\infty} Z_n = \mathbb{I}_B$:

$$\mathbb{E}[Z_{n+1}|\mathcal{A}_{V_n}] = \mathbb{E}[\mathbb{E}[\mathbb{I}_B|\mathcal{A}_{V_{n+1}}]|\mathcal{A}_{V_n}] = \mathbb{E}[\mathbb{E}[\mathbb{I}_B|\mathcal{A}_{V_n}]|\mathcal{A}_{V_{n+1}}] = \mathbb{E}[\mathbb{I}_B|\mathcal{A}_{V_n}] = Z_n\,.$$

Hence $\mathbb{P}(B)^2 = \mathbb{E}[\mathbb{I}_B\mathbb{P}(B)] = \mathbb{E}[\mathbb{I}_B^2] = \mathbb{P}(B)$ and $\mathbb{P}(B) \in \{0, 1\}$. □

Next we introduce some notation for rooted percolation on $\mathbb{T}$:

Notation 70. In the context of rooted percolation and for $v \in V$ we write

$$O_v^\Pi := \{v \leftrightarrow \Pi \cap V(\mathbb{T}^v)\} \qquad \Pi \in \Pi(o) \tag{3.18a}$$

$$O_v := \{v \leftrightarrow \infty\} = \{v \leftrightarrow \partial\mathbb{T}^v\}\,, \tag{3.18b}$$

where those events mean "there is an *open downpath* from w to the cutset Π" and "there is an *open downray* starting at v".

The following lemma allows us to concentrate exclusively on rooted percolation (see [Tem08] for a proof):

Lemma 71. *Let $\mathcal{P} \in \mathcal{C}_p^k(V)$, for finite k. Then*

$$(\exists\, v \in V : \mathbb{P}(O_v) > 0) \Leftrightarrow \mathbb{P}(\mathcal{P} \textit{ percolates on } \mathbb{T}) = 1\,, \tag{3.19a}$$

$$(\forall\, v \in V : \mathbb{P}(O_v) = 0) \Leftrightarrow \mathbb{P}(\mathcal{P} \textit{ percolates on } \mathbb{T}) = 0\,. \tag{3.19b}$$

In the case $k = s = 0$ we can change the $\exists$ to $\forall$ in (3.19a), which is needed in the proof of proposition 87. Finally the obvious relationship between rooted percolation reaching a cutset $\Pi \in \Pi(o)$ or the boundary $\partial\mathbb{T}$ from o is:

$$\forall\, w \in V : \quad O_w = \bigcap_{\Pi\in\Pi(o)} O_w^\Pi\,. \tag{3.20}$$

This holds already for the intersection over an *exhaustive sequence of cutsets* $\{\Pi_m\}_{m\in\mathbb{N}}$, i.e. $\forall\, v \in V : \exists\, m_v \in \mathbb{N} : \exists\, w \in \Pi_{m_v} : v$ is an ancestor of w. A central tool is the following two moment methods:

Lemma 72 (First moment method [LP11, section 5.2])**.** *We have*

$$\mathbb{P}(O_o) = \mathbb{P}(o \leftrightarrow \infty) \leq \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} \mathbb{P}(o \leftrightarrow v)\,. \tag{3.21}$$

Lemma 73 (Weighted second moment method [LP11, section 5.3])**.**

$$\mathbb{P}(O_o) = \mathbb{P}(o \leftrightarrow \infty) \geq \inf_{\Pi \in \Pi(o)} \sup_{\mu \in \mathcal{M}_1(\Pi)} \frac{1}{\mathcal{E}(\mu)}\,, \tag{3.22a}$$

where $\mathcal{M}_1(\Pi)$ *is the set of probability measures on the vertex cutset* Π *and the* energy $\mathcal{E}(\mu)$ *of* $\mu \in \mathcal{M}_1(\Pi)$ *is determined by*

$$\mathcal{E}(\mu) = \sum_{v,w \in \Pi} \mu(v)\mu(w)\kappa(v,w)\,. \tag{3.22b}$$

and κ *is the symmetric* percolation kernel

$$\kappa : \qquad V^2 \to \mathbb{R}_+ \qquad (v,w) \mapsto \kappa(v,w) := \frac{\mathbb{P}(o \leftrightarrow v, o \leftrightarrow w)}{\mathbb{P}(o \leftrightarrow v)\mathbb{P}(o \leftrightarrow w)}\,. \tag{3.22c}$$

3.4.3 Upper bound on $p^{k,s}_{max}(V)$

The task is to establish an upper bound on $p^{k,s}_{max}(V)$. In other words, we want to guarantee percolation for high enough p. The first step in section 3.4.3 is to use the second moment method to translate this problem into the search for a suitable exponential bound on the percolation kernel. Then we use k,s-independence to bound the percolation kernel in terms of a conditional probability along a single downray. Hence we can guarantee percolation as soon as we can bound this conditional probability from below in sufficient exponential terms. The percolation along a single downray is just a Bernoulli random field with parameter p and dependency graph $\mathbb{N}_{(k)}$. In the second step in section 3.4.3 we apply the generic minimality of Shearer's measure and a lower bound on $\mu_{\mathbb{N}_{(k)},p}$ to get such an exponential lower bound of parameter ξ. Finally we relate ξ and $br(\mathbb{T})$ and derive the upper bound.

Percolation kernel estimates

In proposition 74 we state a sufficient condition on the percolation kernel in order to percolate. This condition relates the second moment method to the branching number. In proposition 75 we bound the percolation kernel for k,s-independent percolation in terms of conditional probabilities along a single downray, hence providing a simpler means to derive the sufficient condition in subsequent steps.

Proposition 74. *Let* $\mathcal{P} \in \mathcal{C}^{k,s}_{?,o}(V)$ *and* $\alpha < br(\mathbb{T})$, $C \in \mathbb{R}_+$ *such that* $\forall\, v,w \in V$:

$$\kappa(v,w) \leq C\alpha^{l(v \curlywedge w)}\,, \tag{3.23}$$

then $\mathcal{P}$ *percolates.*

Remark. The “?” in $\mathcal{C}^{k,s}_{?,o}(V)$ means that we place no restriction yet on the marginals of $\mathcal{P}$. The *confluent* of v and w is $v \curlywedge w$. See also figure 3.2.

Proof. Take $\beta \in]\alpha, br(\mathbb{T})[$ and let g be a β-flow. Define $\mu(v) := \frac{g(v)}{g(o)}$, hence $\mu|_\Pi \in \mathcal{M}_1(\Pi)$ for each vertex cutset $\Pi \in \Pi(o)$. We have

$$\begin{aligned}
&\mathcal{E}(\mu|_\Pi) \\
&= \sum_{v,w\in\Pi} \mu|_\Pi(v)\mu|_\Pi(w)\kappa(v,w) \\
&\leq \sum_{v,w\in\Pi} \mu(v)\mu(w)C\alpha^{l(v\curlywedge w)} \\
&= C\sum_{n=0}^{\infty}\alpha^n \sum_{\substack{v,w\in\Pi\\ v\curlywedge w=:u\in L(\mathbb{T},n)}} \mu(v)\mu(w) \\
&\leq C\sum_{n=0}^{\infty}\alpha^n \sum_{u\in L(\mathbb{T},n)} \sum_{\substack{v,w\in\Pi\\ u\in P(o,v\curlywedge w)}} \frac{g(v)g(w)}{g(o)^2} && \text{more nodes} \\
&= \frac{C}{g(o)^2}\sum_{n=0}^{\infty}\alpha^n \sum_{u\in L(\mathbb{T},n)} g(u)^2 && \text{flow property} \\
&\leq \frac{C}{g(o)^2}\sum_{n=0}^{\infty}\left(\frac{\alpha}{\beta}\right)^n \sum_{u\in L(\mathbb{T},n)} g(u) && \beta\text{-flow} \\
&\leq \frac{C}{g(o)}\sum_{n=0}^{\infty}\left(\frac{\alpha}{\beta}\right)^n && \text{flow property} \\
&= \frac{C}{g(o)}\frac{\beta}{\beta-\alpha} && \alpha<\beta,
\end{aligned}$$

which is a finite bound independent of Π. Apply the weighted second moment method (see lemma 73) to see that $\mathbb{P}(o \leftrightarrow \infty) > 0$ and conclude. □

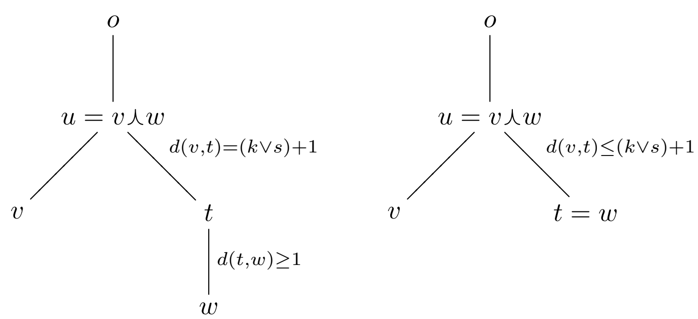

Figure 3.2: Decomposition of the percolation kernel $\kappa(v,w)$ for k,s-independent, rooted site percolation. The node $t \in P(u,w)$ has distance $(k \vee s)+1$ from u if the path $P(u,w)$ is longer than this (left side), otherwise $t = w$ (right side).

Proposition 75. *We use the notation from figure 3.2. Then* $\forall\, k, s \in \mathbb{N}_0, \mathcal{P} \in \mathcal{C}^{k,s}_{p,o}(V), v, w \in V$:

$$\kappa(v,w) \leq \frac{1}{\mathbb{P}(o \leftrightarrow t | t \leftrightarrow w)} \,. \tag{3.24}$$

Proof. We use the notation from figure 3.2. In the case $d(u,w) > (k \vee s) + 1$ we have

$$\begin{aligned}
&\kappa(v,w)\\
&= \frac{\mathbb{P}(o \leftrightarrow v, o \leftrightarrow w)}{\mathbb{P}(o \leftrightarrow v)\mathbb{P}(o \leftrightarrow w)}\\
&= \frac{\mathbb{P}(o \leftrightarrow v, o \leftrightarrow w)}{\mathbb{P}(o \leftrightarrow v)\mathbb{P}(t \leftrightarrow w)\mathbb{P}(o \leftrightarrow t | t \leftrightarrow w)}\\
&= \frac{\mathbb{P}(o \leftrightarrow v, u \leftrightarrow w)}{\mathbb{P}(o \leftrightarrow v, t \leftrightarrow w)} \frac{1}{\mathbb{P}(o \leftrightarrow t | t \leftrightarrow w)} && \text{using } (k \vee s)\text{-independence}\\
&\leq \frac{1}{\mathbb{P}(o \leftrightarrow t | t \leftrightarrow w)} \,.
\end{aligned}$$

In the case $d(u,w) \leq (k \vee s) + 1$ we have $t = w$ and

$$\kappa(v,w) = \frac{\mathbb{P}(o \leftrightarrow v, o \leftrightarrow w)}{\mathbb{P}(o \leftrightarrow v)\mathbb{P}(o \leftrightarrow w)} = \mathbb{P}(u \leftrightarrow t | o \leftrightarrow v) \frac{1}{\mathbb{P}(o \leftrightarrow t)} \leq \frac{1}{\mathbb{P}(o \leftrightarrow t | t \leftrightarrow w)} \,.$$

□

Uniform bound by Shearer's measure

The following proposition combines our knowledge of $\mu_{\mathbb{Z}_{(k)},p}$ and its properties with the simplified condition on the percolation kernel from proposition 75 to ensure uniform percolation.

Proposition 76.

$$\forall\, k, s \in \mathbb{N}_0: \quad p^{k,s}_{max}(V) \leq \begin{cases} g_k(br(\mathbb{T})) & \text{if } br(\mathbb{T}) \leq \frac{k+1}{k} \\ p^{\mathbb{Z}_{(k)}}_{sh} & \text{if } br(\mathbb{T}) \geq \frac{k+1}{k} \,. \end{cases} \tag{3.25}$$

Furthermore for $br(\mathbb{T}) > \frac{k+1}{k}$ *every percolation in* $\mathcal{C}^{k,s}_{p^{\mathbb{Z}_{(k)}}_{sh},o}(V)$ *percolates. In the case* $k = 0$ *we interpret* $\frac{1}{0} := \infty$.

Proof. Let $p \geq p^{\mathbb{Z}_{(k)}}_{sh}$. Use the notation from figure 3.2. Let ξ be the unique solution of the equation $1 - p = \xi(1-\xi)^k$ from (2.52). In a first step we use (3.24), the minimality of Shearer's measure (2.28b), the explicit minoration of Shearer's measure on $\mathbb{N}_{(k)}$ (2.56b) and the fact that $l(t) \leq l(u) + (k \vee s) + 1$ to majorize the percolation kernel as follows:

$$\kappa(v,w) \leq \frac{1}{\mathbb{P}(o \leftrightarrow t | t \leftrightarrow w)} \leq \frac{1}{\mu_{\mathbb{Z}_{(k)},p}(o \leftrightarrow t | t \leftrightarrow w)} \leq \frac{1}{\xi^{l(t)}} \leq \xi^{-(k \vee s)-1} \xi^{-l(u)} \,.$$

In the second step we want to apply the sufficient exponential bound condition on the percolation kernel from proposition 74, hence we have to relate ξ with

$br(\mathbb{T})$. The function g_k (3.11) satisfies $g_k(\frac{1}{\xi}) = p$, has a global minimum in $\frac{k+1}{k}$ with value $p_{sh}^{\mathbb{Z}_{(k)}}$ and induces a strictly monotone decreasing bijection between $[1, \frac{k+1}{k}]$ and $[p_{sh}^{\mathbb{Z}_{(k)}}, 1]$.

Case $br(\mathbb{T}) \leq \frac{k+1}{k}$ and $g_k(br(\mathbb{T})) < p = \xi^k(1-\xi)$: Apply proposition 74 with $C := \xi^{-(k\vee s)-1}$ and $\alpha := \frac{1}{\xi} < br(\mathbb{T})$ to show that we percolate. This proves the g_k part of (3.25).

Case $br(\mathbb{T}) > \frac{k+1}{k}$ and $p_{sh}^{\mathbb{Z}_{(k)}} \leq p$: Apply proposition 74 with $C := \xi^{-(k\vee s)-1}$ and $\alpha := \frac{1}{\xi} \leq \frac{k+1}{k} < br(\mathbb{T})$ to show that we percolate. This proves the $p_{sh}^{\mathbb{Z}_{(k)}}$ part of (3.25) and the percolation statement at $p_{sh}^{\mathbb{Z}_{(k)}}$. □

We show that we need uniformly bounded elsewhere-dependences to guarantee percolation for high p. The counterexample consists of *multiplexing* a distribution indexed by $\mathbb{N}_0$ over the corresponding level of $\mathbb{T}$.

Model 77. For $p \geq p_{sh}^{\mathbb{Z}_{(k)}}$ let $\mathcal{Z} := \{\mathcal{Z}_n\}_{n\in\mathbb{N}_0}$ be a collection of k-independent Bernoulli(p) rvs. Define a site percolation $Z := (Z_v)_{v\in V}$ on the rooted tree $\mathbb{T}$ by

$$Z_v := \mathcal{Z}_{l(v)} . \tag{3.26}$$

Proposition 78. *For every $s \in \mathbb{N}$ we have $Z \notin \mathcal{C}_{p,o}^{k,s}(V)$ and Z percolates iff $p = 1$.*

Proof. All the sites on a chosen level of $\mathbb{T}$ realize a.s. in the same state. Therefore the elsewhere-dependence s_v of v is in the range $2\,l(v) \leq s_v \leq 2\,l(v) + k$ and unbounded in v. Using (3.26) and k-independence we get

$$\forall\, n \in \mathbb{N} : \quad \mathbb{P}(o \leftrightarrow L(\mathbb{T}, n)) = \mathbb{P}(\mathcal{Z}_0 = \ldots = \mathcal{Z}_n = 1) \leq p^{n/(k+1)} .$$

This exponential upper bound implies that $\mathbb{P}(O_o) = 0$ iff $p < 1$. □

3.4.4 Lower bound on $p_{max}^{k,s}(V)$

To derive a lower bound on $p_{max}^{k,s}(V)$ we exhibit appropriate nonpercolating percolation models. The proof of proposition 76 suggests to look for percolations being $\mu_{\mathbb{Z}_{(k)},p}$-distributed along downrays. To be as general as possible we also want $s = 0$. Section 3.4.4 presents a procedure to construct a $k, 0$-independent percolation model with given distribution along downrays. We then apply this construction to probability distributions derived from $\mu_{\mathbb{Z}_{(k)},p}$ and $\mu_{[N]_{(k)},p}$. Applying the first moment method and relating the relevant parameters to $br(\mathbb{T})$ yields the lower bounds.

Tree fission

In this section we show how to create a $k, 0$-independent percolation model from a k-independent Bernoulli random field $\mathcal{Z}$ indexed by $\mathbb{N}_0$. Additionally the resulting model has the same distribution along all downrays, namely the one of $\mathcal{Z}$, and is invariant under automorphisms of the rooted tree. The generic construction is presented in proposition 79 and specialized to our setting in corollary 80.

Proposition 79. *Let $\mathcal{Z} := \{\mathcal{Z}_n\}_{n\in\mathbb{N}_0}$ be a Bernoulli random field and $\mathbb{T} := (V, E)$ be a tree rooted at o. Then there exists a unique probability measure ν, called the* $\mathbb{T}$-fission of $\mathcal{Z}$*, under which the Bernoulli random field $Z := \{Z_v\}_{v\in V}$ has the following properties:*

$$\forall\, W \subseteq V : \quad \textit{if } \forall\, v, w \in W : v \notin V(\mathbb{T}^w)\,, \quad \textit{then the subfields } \{Z_{V(\mathbb{T}^w)}\}_{w\in W} \textit{ are independent.} \tag{3.27a}$$

$$\forall\, v \in V : \quad Z_{P(o,v)} \textit{ has the same law as } \{\mathcal{Z}_{l(w)}\}_{w\in P(o,v)}\,. \tag{3.27b}$$

Furthermore Z is invariant under automorphisms of the rooted tree.

Proof. For $v \in V$ let $A(v) := P(o, v) \setminus \{v\}$ be the set of all ancestors of v. Let $\mathcal{S}$ be the family of vertices of finite connected components of V containing o. For $R \in \mathcal{S}$ define the probability measure ν_R on $\{0,1\}^R$ by setting

$$\forall\, \vec{s}_R \in \{0,1\}^R : \quad \nu_R(Y_R = \vec{s}_R) := \prod_{v\in R} \mathbb{P}(\mathcal{Z}_{l(v)} = s_v | \forall\, w \in A(v) : \mathcal{Z}_{l(w)} = s_w)\,. \tag{3.28}$$

We claim that $\{\nu_R\}_{R\in\mathcal{S}}$ is a consistent family à la Kolmogorov. Furthermore each ν_R has properties (3.27). We admit the claim for the moment and show its proof after finishing the main argument. Hence Kolmogorov's existence theorem [Bil95, theorem 36.2] yields an extension ν of the above family. The probability measure ν fulfils (3.27) because all its marginals ν_R do so. Uniqueness follows from the fact that the properties (3.27) imply the construction of the marginal laws ν_R via (3.28) and the $\pi - \lambda$ theorem [Bil95, theorem 3.3].

We prove the above claim about the consistency and properties of the family $\{\nu_R\}_{R\in\mathcal{S}}$ by induction over the cardinality of R. Let $R, T \in \mathcal{S}$. Then $\forall\, \vec{s}_{R\cup T} \in \{0,1\}^{R\cup T}$:

$$\nu_{R\cup T}(Y_{R\cup T} = \vec{s}_{R\cup T}) = \nu_{R\cap T}(Y_{R\cap T} = \vec{s}_{R\cap T}) \times \left(\prod_{v\in R\setminus T} \mathbb{P}(\mathcal{Z}_{l(v)} = s_v | \forall\, w \in A(v) : \mathcal{Z}_{l(w)} = s_w)\right)\left(\prod_{v\in T\setminus R} \cdots\right).$$

Hence ν_S and ν_T coincide on their common support $\{0,1\}^{R\cap T}$. This implies consistency of the family $\{\nu_R\}_{R\in\mathcal{S}}$.

It remains to show that ν_R is a probability measure on $\{0,1\}^R$ with properties (3.27). We prove this by induction over the cardinality of R. The induction base for $R = \{o\}$ is

$$\nu_{\{o\}}(Y_o = 0) + \nu_{\{o\}}(Y_o = 1) = \mathbb{P}(\mathcal{Z}_0 = 0) + \mathbb{P}(\mathcal{Z}_0 = 1) = 1\,.$$

The induction step reduces R to $T := R \setminus \{v\}$ for some leaf v of $G(R)$. Hence

$$\sum_{\vec{s}_R} \nu_R(Y_R = \vec{s}_R) = \sum_{\vec{s}_T} \sum_{s_v\in\{0,1\}} \nu_R(Y_v = s_v, Y_T = \vec{s}_T)$$

$$= \sum_{\vec{s}_T} \nu_R(Y_T = \vec{s}_T) \underbrace{\sum_{s_v \in \{0,1\}} \mathbb{P}(\mathcal{Z}_{l(v)} = s_v | \forall\, w \in A(v) : \mathcal{Z}_{l(w)} = \vec{s}_w)}_{=1}$$
$$= \sum_{\vec{s}_T} \nu_T(Y_T = \vec{s}_T)$$
$$= 1\,.$$

For independence suppose that $W \subseteq R \in S$ fulfils the condition of (3.27a). Let $U := \bigcup_{w \in W} A(w)$ and for $w \in W$ let $V_w := V(\mathbb{T}^w) \cap R$. Then (3.28) entails that

$$\nu_R(\forall\, w \in W : Z_{V_w} = \vec{s}_{V_w} | Z_U = \vec{s}_U) = \prod_{w \in R} \nu_R(Z_{V_w} = \vec{s}_{V_w} | Z_U = \vec{s}_U)\,.$$

Conditional independence on Z_U implies independence as in (3.27a).

We turn to the distribution along downpaths. For $v \in V$ we have $P := P(o,v) =: \{o =: w_0, \ldots, w_{l(v)} := v\} \in S$. Hence $\forall\, \vec{s}_P \in \{0,1\}^P$:

$$\nu_P(Z_P = \vec{s}_P)$$
$$= \prod_{i=0}^{l(v)} \nu_P(Z_{w_i} = s_{w_i} | Z_{A(w_i)} = \vec{s}_{A(w_i)})$$
$$= \prod_{i=0}^{l(v)} \mathbb{P}(\mathcal{Z}_i = s_{w_i} | \forall\, w \in A(w_i) : \mathcal{Z}_{l(w)} = s_w)$$
$$= \mathbb{P}(\forall\, w \in P : \mathcal{Z}_{l(w)} = s_w)\,.$$

Finally the invariance under automorphisms of the rooted tree is a result of the obliviousness of the construction to the ordering of the children. □

Corollary 80. *If $\mathcal{Z}$ from proposition 79 is k-independent and has marginal parameter p then ν, the $\mathbb{T}$-fission of $\mathcal{Z}$, is the law of a percolation in $\mathcal{C}^{k,0}_{p,o}(V)$ invariant under automorphisms of the rooted tree.*

Proof. The definition of ν implies that it is the law of a rooted site percolation which is invariant under automorphisms of the rooted tree. k-independence and the fact that $\nu(Y_v = 1) = p$ follow from (3.27b), while $s = 0$ follows from the independence over disjoint subtrees in (3.27a). □

The canonical model

For $p \geq p_{sh}^{\mathbb{Z}_{(k)}}$ we derive a $k,0$-independent percolation model from $\mu_{\mathbb{Z}_{(k)},p}$. It does not percolate for small $br(\mathbb{T})$ if p is smaller than the g_k part of (3.25), leading to a lower bound on $p^{k,0}_{max}(V)$.

Model 81. Let $k \in \mathbb{N}$, $p \geq p_{sh}^{\mathbb{Z}_{(k)}}$ and $\mathcal{Z} := \{\mathcal{Z}_n\}_{n \in \mathbb{N}_0}$ be $\mu_{\mathbb{N}_{(k)},p}$-distributed (shifting indices by 1). Define the *canonical model of k-independent site percolation with parameter p*, abbreviated $\mathcal{P}_p^{can(k)}$, as the $\mathbb{T}$-fission of $\mathcal{Z}$.

Remark. We named our canonical model after the canonical model of Balister & Bollobás [BB06]. Their model is a bond percolation model, whose limit case is defined in the following way: for $p \geq \frac{3}{4}$ let $\xi \geq \frac{1}{2}$ be the unique solution of $1-p = \xi(1-\xi)$ (compare with (2.52)). Define the bond percolation $Z := \{Z_e\}_{e\in E}$ by

$$Z_e := 1 - (1 - X_{\mathfrak{p}(v)})X_v\,, \tag{3.29}$$

where $e := (\mathfrak{p}(v), v)$. See also figure 3.3. Hence it has dependency parameters $k = s = 1$. We see that Y_e is closed iff $(X_{\mathfrak{p}(v)}, X_v) = (0,1)$ and, comparing it with (2.54), we deduce that it is $\mu_{\mathbb{Z},p}$-distributed along downrays. Balister & Bollobás do not mention this link explicitly, though. They not only use this model in its role as nonpercolating counterexample for a lower bound on $p^1_{max}(E)$, as we do with our canonical model in proposition 82, but also show that it has the smallest probability to percolate among all percolations in $\mathcal{C}^{1,1}_{p,o}(E)$, their equivalent to our calculations in section 3.4.3.

Balister & Bollobás' explicit construction is easily generalizable to bond models with higher k, but only for $s \geq 2k-1$. Furthermore their inductive approach fails us already for $k \geq 2$. Thus its main inspiration has been to look for $k,0$-independent percolation models being $\mu_{\mathbb{Z}_{(k)},p}$-distributed along all downrays, leading to the tree-fission and our construction in model 81.

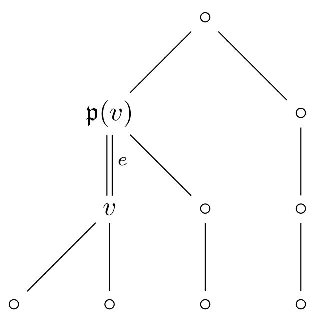

Figure 3.3: Construction of Balister & Bollobás' canonical model. See (3.29).

Proposition 82. *For all* $k \in \mathbb{N} : \mathcal{P}^{can(k)}_p \in \mathcal{C}^{k,0}_{p,o}(V)$. *If* $br(\mathbb{T}) \leq \frac{k+1}{k}$ *and* $p \in \left[p^{\mathbb{Z}_{(k)}}_{sh}, g_k(br(\mathbb{T}))\right[$, *then* $\mathcal{P}^{can(k)}_p$ *does not percolate. This implies that*

$$\forall\, k \in \mathbb{N},\, br(\mathbb{T}) \in \left[1, \frac{k+1}{k}\right[\;:\quad p^{k,0}_{max}(V) \geq g_k(br(\mathbb{T}))\,. \tag{3.30}$$

Proof. As $\mathcal{Z}$ from model 81 is k-independent and has marginal parameter p corollary 80 asserts that $\mathcal{P}^{can(k)}_p \in \mathcal{C}^{k,0}_{p,o}(V)$.

Remember that $p < g_k(br(\mathbb{T}))$ is equivalent to $\xi < \frac{1}{br(\mathbb{T})}$, hence we can choose $\varepsilon > 0$ such that $(1+\varepsilon)\xi < \frac{1}{br(\mathbb{T})}$. The first moment method (lemma 72) yields

$$\mathbb{P}(o \leftrightarrow \infty)$$

$$
\begin{aligned}
&\leq \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} \mathbb{P}(o \leftrightarrow v) \\
&\leq \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} C[(1+\varepsilon)\xi]^{l(v)+1} && \text{by (2.56c)} \\
&= C(1+\varepsilon)\xi \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} \left[\frac{1}{(1+\varepsilon)\xi}\right]^{-l(v)} \\
&= 0 && \text{by definition of } br(\mathbb{T}) \text{ in (3.4).}
\end{aligned}
$$

Therefore $\mathcal{P}_p^{can(k)}$ does not percolate and (3.30) follows directly. □

The cutup model

For $N \in \mathbb{N}$ and $p_{sh}^{[N]_{(k)}} < p_{sh}^{\mathbb{Z}_{(k)}}$ we derive a $k, 0$-independent percolation model from $\mu_{[N]_{(k)}, p_{sh}^{[N]_{(k)}}}$. It never percolates. In the limit $N \to \infty$ this yields a lower bound of $p_{sh}^{\mathbb{Z}_{(k)}}$ for $p_{max}^{k,0}(V)$.

Model 83. Let $k, N \in \mathbb{N}$ and $\mathcal{Z} := \{\mathcal{Z}_n\}_{n \in \mathbb{N}_0}$ have distribution equivalent to independent copies of $\mu_{[N]_{(k)}, p_{sh}^{[N]_{(k)}}}$ on $\{mN, mN+1, \dots, (m+1)N-1\}$ for all $m \in \mathbb{N}_0$. Define the *N-cutup model of k-independent site percolation*, abbreviated $\mathcal{P}^{cut(k,N)}$, as the $\mathbb{T}$-fission of $\mathcal{Z}$.

Proposition 84. *For all $k, N \in \mathbb{N} : \mathcal{P}^{cut(k,N)} \in \mathcal{C}^{k,0}_{p_{sh}^{[N]_{(k)}}, o}(V)$. It has percolation cluster diameters a.s. bounded by $4N-4$. Hence it does not percolate. This implies that*

$$
\forall\, k \in \mathbb{N}: \quad p_{max}^{k,0}(V) \geq p_{sh}^{\mathbb{Z}_{(k)}}\,. \tag{3.31}
$$

Remark. It is possible to generate models like the cutup model for every $p < p_{sh}^{\mathbb{Z}_{(k)}}$ [She85, proof of theorem 1].

Proof. As $\mathcal{Z}$ from model 83 is k-independent and has marginal parameter p corollary 80 asserts that $\mathcal{P}^{cut(k,N)} \in \mathcal{C}^{k,0}_{p_{sh}^{[N]_{(k)}}, o}(V)$.

To bound cluster diameters note that $\mu_{[N]_{(k)}, p_{sh}^{[N]_{(k)}}}$ blocks going more than $2N-2$ steps up or down along a downray. Hence cluster diameters are a.s. bounded by $4N-4$ and $\mathcal{P}^{cut(k,N)}$ does not percolate. Thus $p_{max}^{k,0}(V) \geq p_{sh}^{[N]_{(k)}}$. Finally we know from (2.53) that $p_{sh}^{[N]_{(k)}} \xrightarrow[N\to\infty]{} p_{sh}^{\mathbb{Z}_{(k)}}$. □

3.4.5 Determining $p_{min}^{k,s}(V)$

To determine $p_{min}^{k,s}(V)$ we take the opposite approach from $p_{max}^{k,s}(V)$. For a uniform lower bound we use the first moment method in proposition 85 on percolations with small enough p. An upper bound follows from the so-called minimal model 86, again built by tree-fission from section 3.4.4. We show that it percolates for sufficiently high p employing the sufficient conditions on the percolation kernel from section 3.4.3, effectively using the second moment method.

Proposition 85.

$$\forall\, k \in \mathbb{N}_0, s \in \mathbb{N}_0 \uplus \{\infty\}: \quad p^{k,s}_{min}(V) \geq \frac{1}{br(\mathbb{T})^{k+1}}\,. \tag{3.32}$$

Proof. Let $\mathcal{P} \in \mathcal{C}^{k,s}_{p,o}(V)$ with $p < \frac{1}{br(\mathbb{T})^{k+1}}$. Then the first moment method (lemma 72) results in

$$\begin{aligned}
&\mathbb{P}(o \leftrightarrow \infty) \\
&\leq \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} \mathbb{P}(o \leftrightarrow v) \\
&\leq \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} p^{\left\lceil \frac{l(v)}{k+1} \right\rceil} && k\text{-independence along downrays} \\
&\leq \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} \left(p^{-\frac{1}{k+1}}\right)^{-(k+1)\left\lceil \frac{l(v)}{k+1} \right\rceil} \\
&\leq \inf_{\Pi \in \Pi(o)} \sum_{v \in \Pi} \left(p^{-\frac{1}{k+1}}\right)^{-l(v)} && \text{as } (k+1)\left\lceil \frac{l(v)}{k+1} \right\rceil > l(v) \\
&= 0 && \text{as } br(\mathbb{T}) < p^{-\frac{1}{k+1}}\,.
\end{aligned}$$

Hence $\mathcal{P}$ does not percolate and (3.32) follows trivially. □

Model 86. Let $X := \{X_n\}_{n \in \mathbb{N}_0}$ be an i.i.d. Bernoulli random field with parameter $\hat{p} := p^{1/(k+1)}$. Define $\mathcal{Z} := \{\mathcal{Z}_n\}_{n \in \mathbb{N}_0}$ by $\forall\, n \in \mathbb{N}_0 : \mathcal{Z}_n := \prod_{i=0}^{k} X_{n+i}$. Define the *minimal model of k-independent site percolation with parameter p*, abbreviated $\mathcal{P}^{min(k)}_p$, as the $\mathbb{T}$-fission of $\mathcal{Z}$.

Proposition 87. *For all $k \in \mathbb{N} : \mathcal{P}^{min(k)}_p \in \mathcal{C}^{k,0}_{p,o}(V)$. If $p > \frac{1}{br(\mathbb{T})^{k+1}}$, then $\mathcal{P}^{min(k)}_p$ percolates, which entails that*

$$\forall\, k \in \mathbb{N}: \quad p^{k,0}_{min}(V) \leq \frac{1}{br(\mathbb{T})^{k+1}}\,. \tag{3.33}$$

Proof. As $\mathcal{Z}$ from model 86 is k-independent and has marginal parameter p corollary 80 asserts that $\mathcal{P}^{min(k)}_p \in \mathcal{C}^{k,0}_{p^{[N](k)}_{sh},o}(V)$.

Let $Z := \{Z_v\}_{v \in V}$ be $\mathcal{P}^{min(k)}_p$-distributed and $p > \frac{1}{br(\mathbb{T})^{k+1}}$. Looking at model 86, we see that $\mathbb{P}(\mathcal{Z}_{[n]} = \vec{1}) = \mathbb{P}(X_{[n+k]} = \vec{1}) = \hat{p}^{n+k}$, with $\hat{p} = p^{1/(k+1)}$. Use the notation from figure 3.2 and apply the bound on the percolation kernel (3.24) to arrive at:

$$\kappa(v,w) \leq \frac{1}{\mathbb{P}(o \leftrightarrow t | t \leftrightarrow w)} \leq \frac{1}{\hat{p}^{l(t)}} \leq \hat{p}^{-k-1}\hat{p}^{-l(u)}$$

Apply proposition 74 with $C := \hat{p}^{-k-1}$ and $\alpha := \frac{1}{\hat{p}} < br(\mathbb{T})$ to show that we percolate. This proves (3.33). □

3.4.6 The connection with quasi-independence

In this section we show that in both cases (propositions 76 and 87) where we apply the second moment method via exponential bounds on the percolation kernel our k,s-independent percolations are also quasi-independent (3.6). This gives an a posteriori connection with Lyons' work and explains why we have been able to exploit percolation kernels so effectively.

Proposition 88. *Let $p > p_{sh}^{\mathbb{Z}_{(k)}}$. Then $\forall\,\mathcal{P} \in \mathcal{C}_{p,o}^{k;s}(V), \forall\, v, w \in V$:*

$$\kappa(v,w) \leq \frac{\xi^{k-(k\vee s)}}{(k+1)\xi - k} \times \frac{1}{\mathbb{P}(o \leftrightarrow u)}\,, \tag{3.34}$$

hence $\mathcal{P}$ is quasi-independent.

Remark. It is an artefact of our use of (3.24) that we can not show (3.34) to hold for $p = p_{sh}^{\mathbb{Z}_{(k)}}$, where $\xi = \frac{k}{k+1}$. As we believe this artefact to be genuine, we conjecture that quasi-independence does not hold for $\mathcal{P}^{can(k)}_{p_{sh}^{\mathbb{Z}_{(k)}}}$.

Proof. Let $p > p_{sh}^{\mathbb{Z}_{(k)}}$. We use the notation from figure 3.2. Then the minimality of Shearer's measure (2.28a), the explicit minoration on $\mathbb{Z}_{(k)}$ in (2.56a) and the fact that $l(t) \leq l(u) + (k \vee s) + 1$ imply that

$$\begin{aligned}
&\mathbb{P}(o \leftrightarrow t | t \leftrightarrow w)\\
&= \mathbb{P}(u \leftrightarrow t | o \leftrightarrow u, t \leftrightarrow w)\mathbb{P}(o \leftrightarrow u | t \leftrightarrow w)\\
&= \mathbb{P}(u \leftrightarrow t | o \leftrightarrow u, t \leftrightarrow w)\mathbb{P}(o \leftrightarrow u)\\
&\geq \mu_{\mathbb{Z}_{(k)},p}(Y_{\{l(u)+1,\dots,l(t)-1\}} = \vec{1} | Y_{\{0,\dots,l(u)\}} = \vec{1}, Y_{\{l(t),\dots,l(w)\}} = \vec{1})\mathbb{P}(o \leftrightarrow u)\\
&\geq \left[\prod_{i=1}^{k} f_k(i)\right] f_k(0)^{(k\vee s)-k}\mathbb{P}(o \leftrightarrow u)\\
&= [(k+1)\xi - k]\,\xi^{(k\vee s)-k}\mathbb{P}(o \leftrightarrow u)\,.
\end{aligned}$$

Together with the bound on k,s-independent percolation kernels (3.24) on $\kappa(v,w)$ this yields (3.34) and quasi-independence. □

Proposition 89. *The minimal percolation model $\mathcal{P}_p^{min(k)}$ is quasi-independent.*

Proof. We use the notation from figure 3.2. The explicit construction in model 86 with $\hat{p} = p^{1/(k+1)}$ and the fact that $l(t) \leq l(u) + k + 1$ imply that

$$\mathbb{P}(o \leftrightarrow t | t \leftrightarrow w) = \hat{p}^{l(t)} \geq \hat{p}^{l(u)+k+1} = \mathbb{P}(o \leftrightarrow u)\,.$$

Together with the bound on k,s-independent percolation kernels (3.24) we get quasi-independence

$$\kappa(v,w) \leq \frac{1}{\mathbb{P}(o \leftrightarrow t | t \leftrightarrow w)} \leq \frac{1}{\mathbb{P}(o \leftrightarrow u)}\,.$$

□

3.4.7 A comment on stochastic domination

Recall that a percolation X *stochastically dominates* a percolation Y iff there is a coupling of X and Y such that $\mathbb{P}(X \geq Y) = 1$. Here the natural order is the partial component-wise order on $\{0,1\}^E$. We show that for $k \geq 1$ our bounds do not imply stochastic domination of an independent percolation by all k-independent percolations for high enough p.

Proposition 90. $\forall\, k \geq 1, p \in [0,1[, b \in [1,\infty[: \exists\, \hat{p} \in [p,1[$ *and* $\mathbb{T}$ *with* $br(\mathbb{T}) = b$ *and a* k*-independent site percolation* Z *on* $\mathbb{T}$ *with parameter* $\hat{p}$ *such that* Z *stochastically dominates only the trivial Bernoulli product field.*

Remark. It is possible to extend proposition 90 to all $(\hat{p}, b) \in [0,1[\times[1,\infty[$, using [She85, proof of theorem 1].

Proof. Denote the d-regular tree by $\mathbb{T}_d$. We know that $p_{sh}^{\mathbb{T}_d} = 1 - \frac{(d-1)^{(d-1)}}{d^d}$ [She85, theorem 2]. Choose d such that $p_{sh}^{\mathbb{T}_d} > p$. By the definition of $p_{sh}^{\mathbb{T}_d}$ (2.38b) there is a finite subtree $\hat{\mathbb{T}}$ of $\mathbb{T}$ with

$$p < \hat{p} := p_{sh}^{\hat{\mathbb{T}}} < p_{sh}^{\mathbb{T}} .$$

Root $\hat{\mathbb{T}}$ at some vertex $\hat{o}$. Replace every edge of $\hat{\mathbb{T}}$ by a length $(k+1)$ path. Add an extra path of $(k+1)$ edges at $\hat{o}$ with endpoint $\bar{o}$. Extend this finite tree further to some arbitrary infinite tree $\bar{\mathbb{T}}$ with branching number b and root it at $\bar{o}$.

For every length $(k+1)$ path in the previous paragraph take its last edge and denote their union by S. Place $\mu_{\hat{\mathbb{T}},\hat{p}}$ on S and fill up the other edges with i.i.d. Bernoulli($\hat{p}$) variables independently of $\mu_{\mathbb{T}',\hat{p}}$ on S. The resulting percolation is $k,0$-independent. By (2.38a) $\mu_{\hat{\mathbb{T}},\hat{p}}$ fulfils $\mu_{\hat{\mathbb{T}},\hat{p}}(Y_{V(\hat{\mathbb{T}})} = \vec{1}) = 0$ and hence the subpercolation on S dominates only the trivial Bernoulli product field. $\square$

Chapter 4

Stochastic domination of Bernoulli product fields

4.1 Introduction

The question under which conditions a Bernoulli random field (short BRF) stochastically dominates a Bernoulli product field (short BPF) is of interest in probability and percolation theory. Knowledge of this kind allows the transfer of results from the independent case to more general settings. Of particular interest are BRFs with a dependency structure described by a graph G and prescribed common marginal parameter p, as they often arise from rescaling arguments [Gri99], dependent models [BR06] or particle systems [Lig05]. In this setting an interesting question is to find lower bounds on p which guarantee stochastic domination for every such BRF.

This question has been investigated in the setting of boot-strap percolation [And93, section 2] and supercritical Bernoulli percolation [AP96, section 2]. Finally Liggett, Schonmann & Stacey [LSS97] derived a generic lower bound in the case of a uniformly bounded graph. Of particular interest is the k-fuzz of $\mathbb{Z}$ (that is $\mathbb{Z}$ with additional edges between all vertices at distance less than k), which is the dependency graph of k-dependent BRFs on $\mathbb{Z}$. In this case they determined the minimal p for which stochastic domination of a non-trivial BPF holds for each such BRF on the k-fuzz of $\mathbb{Z}$. Even more, they have also shown that in this case the parameter of the dominated BPF is uniformly bounded from below and nonzero for this minimal p and made a conjecture about the size of the jump of the value of the parameter of the dominated BPF at this minimal p.

Their main tools have been a sufficient condition highly reminiscent of the Lovász Local Lemma [EL75] (short LLL, also known as the Dobrushin condition [Dob96b] in statistical mechanics) and the explicit use of Shearer's measure [She85] on the k-fuzz of $\mathbb{Z}$ to construct a series of probability measures dominating only trivial BPFs. Recall that Shearer's measure is the optimal boundary case for the LLL. It is also related to the grand canonical partition function of a lattice gas with both hard-core interaction and hard-core self-repulsion [SS05, BFPS11].

Extending the work of Liggett, Schonmann & Stacey in a natural way we demonstrate that the use of Shearer's measure and the overall similarity between their proof and those concerning only Shearer's measure is not coincidence, but part of a larger picture. We show that there is a non-trivial uniform lower bound on the parameter vector of the BPF dominated by a BRF with marginal parameter vector $\vec{p}$ and dependency graph G iff Shearer's measure with prescribed marginal parameter vector $\vec{p}$ exists on G.

After reparametrization the set of admissible vectors $\vec{p}$ is equivalent to the polydisc of absolute and uniform convergence of the cluster expansion of the partition function of a hard-core lattice gas around fugacity $\vec{0}$ [SS05, BFPS11] allowing a high-temperature expansion [Dob96b]. This connection opens the door to a reinterpretation of results from cluster expansion techniques [GK71, FP07, BFP10] or tree equivalence techniques [SS05, sections 6 & 8], leading to improved estimates on admissible $\vec{p}$ for the domination problem. Possible future lines of research include the search for probabilistic interpretations of these combinatorial and analytic results.

The layout of this paper is as follows: we formulate the stochastic domination problem in section 4.2. Section 4.3 contains our new results, followed by examples of reinterpreted bounds in section 4.3.1. Finally section 4.5 deals with the weak invariant case and we refute the conjecture by Liggett, Schonmann & Stacey concerning the minimality of Shearer's measure for the dominated parameter in section 4.6.

4.2 Setup and problem statement

Let $G := (V, E)$ be a locally finite graph. Denote by $\mathcal{N}(v)$ the *set of neighbours* of v and by $\mathcal{N}_1(v) := \mathcal{N}(v) \uplus \{v\}$ the *neighbourhood of v including v itself*. For every subset H of vertices and/or edges of G denote by $V(H)$ the *vertices induced by H* and by $G(H)$ the *subgraph of G induced by H*.

Vectors are indexed by V, i.e. $\vec{x} := (x_v)_{v\in V}$. Scalar operations on vectors act coordinate-wise (as in $\vec{x}\,\vec{y}$) and scalar comparisons hold for all corresponding coordinates of the affected vectors (as in $\vec{0} < \vec{x}$). For $W \subseteq V$ let $\vec{x}_W := (x_v)_{v\in W}$, where needed for disambiguation. We otherwise ignore superfluous coordinates. If we use a scalar x in place of a vector $\vec{x}$ we mean to use $\vec{x} = x\vec{1}$ and call this the *homogeneous setting*. We *always assume the relation* $q = 1 - p$, also in vectorized form and when having corresponding subscripts. Denote by $\mathcal{X}_V := \{0,1\}^V$ the compact *space of binary configurations* indexed by V. Equip $\mathcal{X}_V$ with the natural partial order induced by $\vec{x} \leq \vec{y}$ (isomorph to the partial order induced by the subset relation in $\mathcal{P}(V)$).

A *Bernoulli random field* (short BRF) $Y := (Y_v)_{v\in V}$ on G is a rv taking values in $\mathcal{X}_V$, seen as a collection of Bernoulli rvs Y_v indexed by V. A *Bernoulli product field* (short BPF) X is a BRF where $(X_v)_{v\in V}$ is a collection of independent Bernoulli rvs. We write its law as $\Pi^V_{\vec{x}}$, where $x_v := \Pi^V_{\vec{x}}(X_v = 1)$.

A subset A of the space $\mathcal{X}_V$ or the space $[0,1]^V$ is an *up-set* iff

$$\forall \vec{x} \in A, \vec{y} \in \mathcal{X}_V : \quad \vec{x} \leq \vec{y} \Rightarrow \vec{y} \in A\,.$$

Replacing $\leq$ by $\geq$ yields a *down-set*.

We recall the definition of *stochastic domination* [Lig05]. Let Y and Z be two BRFs on G. Denote by $\mathrm{Mon}(V)$ the set of *monotone continuous functions* from $\mathcal{X}_V$ to $\mathbb{R}$, that is $\vec{s} \leq \vec{t}$ implies $f(\vec{s}) \leq f(\vec{t})$. We say that Y dominates Z stochastically iff they respect monotonicity in expectation:

$$Y \overset{st}{\geq} Z \Leftrightarrow \Big(\forall f \in \mathrm{Mon}(V) : \mathbb{E}[f(Y)] \geq \mathbb{E}[f(Z)]\Big)\,. \tag{4.1}$$

Equation (4.1) actually refers to the laws of Y and Z. We abuse notation and treat a BRF and its law as interchangeable.

For a BRF Y we denote the *set of all dominated Bernoulli parameter vectors* (short: set of dominated vectors) by

$$\Sigma(Y) := \{\vec{c} : Y \overset{st}{\geq} \Pi^V_{\vec{c}}\}\,. \tag{4.2a}$$

It describes all the different BPFs minorating Y stochastically. The set $\Sigma(Y)$ is a closed down-set. The definition of dominated vector extends to a non-empty class C of BRFs by

$$\Sigma(C) := \bigcap_{Y \in C} \Sigma(Y) = \{\vec{c} : \forall Y \in C : Y \overset{st}{\geq} \Pi^V_{\vec{c}}\}\,. \tag{4.2b}$$

For a class C of BRFs denote by $C(\vec{p})$ the subclass consisting of BRFs with marginal parameter vector $\vec{p}$. We call a BPF with law $\Pi^V_{\vec{c}}$, respectively the vector $\vec{c}$, *non-trivial* iff $\vec{c} > 0$. Our *main question* is under which conditions all BRFs in a class C dominate a non-trivial BPF. Even stronger, we ask whether they all dominate a common non-trivial BPF. Hence, given a class C, we investigate the *set of parameter vectors guaranteeing non-trivial domination*

$$\mathcal{P}^C_{dom} := \Big\{\vec{p} \in [0,1]^V : \forall Y \in C(\vec{p}) : \exists \vec{c} > \vec{0} : \vec{c} \in \Sigma(Y)\Big\} \tag{4.2c}$$

and the *set of parameter vectors guaranteeing uniform non-trivial domination*

$$\mathcal{P}^C_{udom} := \Big\{\vec{p} \in [0,1]^V : \exists \vec{c} > \vec{0} : \vec{c} \in \Sigma(C(\vec{p}))\Big\}\,. \tag{4.2d}$$

We have the obvious inclusion

$$\mathcal{P}^C_{udom} \subseteq \mathcal{P}^C_{dom}\,. \tag{4.2e}$$

The main contribution of this paper is the characterization and description of certain properties of the sets (4.2d) and (4.2c) for some classes of BRFs.

The main classes of BRFs we investigate are the *weak and strong dependency classes* $\mathcal{C}^{\mathrm{weak}}_G(\vec{p})$ and $\mathcal{C}^{\mathrm{strong}}_G(\vec{p})$ from (2.3) in section 2.1.1 respectively.

4.3 Main results and discussion

Our main result is

Theorem 91. *For every locally finite graph G we have*

$$\mathcal{P}_{dom}^{\mathcal{C}_G^{weak}} = \mathcal{P}_{udom}^{\mathcal{C}_G^{weak}} = \mathcal{P}_{dom}^{\mathcal{C}_G^{strong}} = \mathcal{P}_{udom}^{\mathcal{C}_G^{strong}} = \mathring{\mathcal{P}}_{sh}^{G}\,. \tag{4.3}$$

Its proof is in section 4.4. Theorem 91 consists of two a priori unrelated statements: The first one consists of the left three equalities in (4.3): uniform and non-uniform domination of a non-trivial BPF are the same, and even taking the smaller class $\mathcal{C}_G^{\text{strong}}$ does not admit more $\vec{p}$. The second one is that these sets are equivalent to the set of parameters for which Shearer's measure exists. The minimality of Shearer's measure (see theorem 33) lets us construct BRFs dominating only trivial BPFs for $\vec{p} \notin \mathring{\mathcal{P}}_{sh}^{G}$ (see section 4.4.2) and clarifies the role Shearer's measure played as a counterexample in the work of Liggett, Schonmann & Stacey [LSS97, section 2]. Even more, this minimality implies:

Theorem 92. *For $\vec{p} \in \mathring{\mathcal{P}}_{sh}^{G}$ define the non-trivial vector $\vec{c}$ component-wise by*

$$c_v := \begin{cases} 1 & \text{if } p_v = 1 \\ 1 - (1 - \Xi_{G_v}(\vec{p}))^{1/|V|} & \text{if } p_v < 1 \text{ and } G_v \text{ is finite} \\ q_v \min\{q_w : w \in \mathcal{N}(v) \cap V(G_v)\} & \text{if } p_v < 1 \text{ and } G_v \text{ is infinite}, \end{cases} \tag{4.4}$$

where G_v is the connected component of v in the subgraph of G induced by all vertices v with $p_v < 1$. Then $\vec{0} < \vec{c} \in \Sigma(\mathcal{C}_G^{weak}(\vec{p}))$.

The proof of theorem 92 is in section 4.4.3. For infinite, connected G we have a *discontinuous transition* in $\vec{c}$ as $\vec{p}$ approaches the boundary of $\mathring{\mathcal{P}}_{sh}^{G}$ (third line of (4.4)), while in the finite case it is continuous (second line of (4.4)). On the other hand there are classes of BRFs having a continuous transition also in the infinite case, for example the class of 2-factors on $\mathbb{Z}$ [LSS97, theorem 3.0].

Our proof trades accuracy in capturing all of $\mathring{\mathcal{P}}_{sh}^{G}$ against accuracy in the lower bound for the parameter of the dominated BPF. It is an intuitive fact (4.9) that $\Sigma(\mathcal{C}_G^{\text{weak}}(\vec{p}))$ should increase with $\vec{p}$, but our explicit lower bound (4.4) decreases in $\vec{p}$. There is an explicit growing lower bound already shown by Liggett, Schonmann & Stacey [LSS97, corollary 1.4], although only on a restricted set of parameters (as in theorem 40).

Equation (2.28b) does not imply that $\mu_{G,\vec{p}} \overset{st}{\leq} Y$ for all $Y \in \mathcal{C}_G^{\text{weak}}(\vec{p})$: for a finite $W \subsetneq V$ take $f := 1 - \mathbb{I}_{\{\vec{0}\}} \in \text{Mon}(W)$ and see that $\Pi_{\vec{p}}^{W} \overset{st}{\ngeq} \mu_{G(W),\vec{p}}$. Furthermore $\Sigma(\mu_{G,\vec{p}})$ is neither minimal nor maximal (with respect to set inclusion) in the class $\mathcal{C}_G^{\text{weak}}(\vec{p})$. The maximal law is $\Pi_{\vec{p}}^{W}$ itself, as $[\vec{0}, \vec{p}] = \Sigma(\Pi_{\vec{p}}^{W})$. We give a counterexample to the minimality of $\Sigma(\mu_{G,\vec{p}})$ in section 4.6.

4.3.1 Reinterpretation of bounds

Theorem 91 allows the application of criterions for admissible $\vec{p}$ for $\mathcal{P}_{udom}^{\mathcal{C}_G^{\text{weak}}}$ to $\mathring{\mathcal{P}}_{sh}^{G}$ and vice-versa. Hence we can play questions about the existence of a BRF dominating only trivial BPFs or the existence of Shearer's measure back and forth. In the following we list known necessary or sufficient conditions for $\vec{p}$ to lie in $\mathring{\mathcal{P}}^G_{sh}$, most of them previously unknown for the domination problem.

The classical sufficient condition is the *Lovász Local Lemma* [EL75] in theorem 34. Homogeneous versions are listed in section 2.4. More recent sufficient conditions by Fernández & Procacci [FP07, BFPS11] based on cluster expansion techniques are in section 2.5.2. An example of a necessary condition by Scott & Sokal [SS05] is given in theorem 180 in section 6.3.2.

4.4 Proofs

We prove theorem 91 by showing all inclusions outlined in figure 4.1. The four centre inclusions follow straight from (4.2e) and (2.3c). The core part are two inclusions marked (UD) and (ND) in figure 4.1. The second inclusion (ND) generalizes an idea of Liggett, Schonmann & Stacey in section 4.4.2. The key is the usage of Shearer's measure on finite subgraphs H for suitable $\vec{p} \in \partial\mathcal{P}^H_{sh}$ to create BRFs dominating only trivial BPFs. Our novel contribution is the inclusion (UD). It replaces the LLL style proof for restricted parameters employed in [LSS97, proposition 1.2] by an optimal bound reminiscent of the optimal bound presented in [SS05, section 5.3], using the fundamental identity (2.11) to full extent. Using preliminary work on Shearer's measure from section 2.2.4, we prove the inclusion (UD) in section 4.4.3.

$$
\begin{array}{ccccccc}
\mathring{\mathcal{P}}^G_{sh} & \overset{\text{(UD)}}{\subseteq} & \mathcal{P}^{\mathcal{C}^{\text{weak}}_G}_{udom} & \subseteq & \mathcal{P}^{\mathcal{C}^{\text{weak}}_G}_{dom} & & \\
 & & \cap\!| & & \cap\!| & & \\
 & & \mathcal{P}^{\mathcal{C}^{\text{strong}}_G}_{udom} & \subseteq & \mathcal{P}^{\mathcal{C}^{\text{strong}}_G}_{dom} & \overset{\text{(ND)}}{\subseteq} & \mathring{\mathcal{P}}^G_{sh}
\end{array}
$$

Figure 4.1: Inclusions in the proof of (4.3).

4.4.1 Tools for stochastic domination

In this section we list useful statements related to stochastic domination between BRFs. They are used as building blocks in the subsequent sections.

Lemma 93 ([Lig05, chapter II, page 79])**.** *Let Y, Z be two BRFs indexed by V, then*

$$
Y \overset{st}{\geq} Z \quad \Leftrightarrow \quad \left(\forall \text{ finite } W \subseteq V : Y_W \overset{st}{\geq} Z_W\right) . \tag{4.5}
$$

Note to self: One way to prove lemma 93 is using the Lemma of Zorn on Kolmogorov consistent chains in the space of all coupling measures (in the context of the theorem by Liggett). ◊

We build on the following technical result, inspired by [Rus82, lemma 1].

Proposition 94. *If $Z := \{Z_n\}_{n\in\mathbb{N}}$ is a BRF with*

$$\forall\, n \in \mathbb{N}, \vec{s}_{[n]} \in \mathcal{X}_{[n]} : \quad \mathbb{P}(Z_{n+1} = 1 | Z_{[n]} = \vec{s}_{[n]}) \geq p_n\,, \tag{4.6}$$

then there exists a $\Pi_{\vec{p}}^{\mathbb{N}}$-distributed X such that $Z \overset{st}{\geq} X$.

Definition 95. For $W \subset V$ and $\vec{s}_W \in \mathcal{X}_W$ we define the *cylinder set* $\Pi_W^{-1}(\vec{s}_W)$ by

$$\Pi_W^{-1}(\vec{s}_W) := \{\vec{t} \in \mathcal{X}_V : \quad \vec{t}_W = \vec{s}_W\}\,. \tag{4.7}$$

Proof. We follow the proof strategy of [Rus82, lemma 1]. We show that ν fulfils the conditions of (4.5). During this proof we interpret $[0]$ as $\emptyset$. We define a probability measure ν on $\mathcal{X}_{\mathbb{N}^2}$ inductively by:

$$\begin{aligned}
&\forall\, n \geq 1, \forall\, \vec{s}_{[n-1]}, \vec{t}_{[n-1]} \in \mathcal{X}_{[n-1]}, \forall\, a, b \in \{0,1\} : \\
&\qquad \nu(\Pi_{\{n\}}^{-1}(a) \times \Pi_{\{n\}}^{-1}(b) \,|\, \Pi_{[n-1]}^{-1}(\vec{s}_{[n-1]}) \times \Pi_{[n-1]}^{-1}(\vec{t}_{[n-1]})) \\
&\qquad\qquad := \begin{cases} = \mathbb{P}(Z_n = 1 | Z_{[n-1]} = \vec{s}_{[n-1]}) & \text{if } (a,b) = (1,1) \\ = 0 & \text{if } (a,b) = (1,0) \\ = p_n - \mathbb{P}(Z_n = 1 | Z_{[n-1]} = \vec{s}_{[n-1]}) & \text{if } (a,b) = (0,1) \\ = 1 - p_n & \text{if } (a,b) = (0,0)\,. \end{cases}
\end{aligned}$$

A straightforward induction over n shows that ν is a probability measure. The induction base is

$$\sum_{s_1, t_1} \nu(\Pi_{\{1\}}^{-1}(s_1) \times \Pi_{\{1\}}^{-1}(t_1)) = (1 - p_1) + (p_1 - \mathbb{P}(Z_1 = 1)) + 0 + \mathbb{P}(Z_1 = 1) = 1\,.$$

The induction step is

$$\begin{aligned}
&\sum_{\vec{s}_{[n]}, \vec{t}_{[n]}} \nu(\Pi_{[n]}^{-1}(\vec{s}_{[n]}) \times \Pi_{[n]}^{-1}(\vec{t}_{[n]})) \\
&= \sum_{\vec{s}_{[n-1]}, \vec{t}_{[n-1]}} \nu(\Pi_{[n]}^{-1}(\vec{s}_{[n]}) \times \Pi_{[n]}^{-1}(\vec{t}_{[n]})) \\
&\underbrace{\times \left(\sum_{s_n, t_n} \nu(\Pi_{\{n\}}^{-1}(s_n) \times \Pi_{\{n\}}^{-1}(t_n) \,|\, \Pi_{[n-1]}^{-1}(\vec{s}_{[n-1]}) \times \Pi_{[n-1]}^{-1}(\vec{t}_{[n-1]})) \right)}_{=1 \text{ by definition of } \nu} \\
&= \underbrace{\sum_{\vec{s}_{[n-1]}, \vec{t}_{[n-1]}} \nu(\Pi_{[n]}^{-1}(\vec{s}_{[n]}) \times \Pi_{[n]}^{-1}(\vec{t}_{[n]}))}_{=1 \text{ by induction}}\,.
\end{aligned}$$

Next we calculate its marginals. Let $n \geq 1$ and $\vec{s}_{[n]} \in \mathcal{X}_{[n]}$. Then we have

$$\begin{aligned}
&\nu(\Pi_{[n]}^{-1}(\vec{s}_{[n]}) \times \mathcal{X}_{\mathbb{N}}) \\
&= \prod_{i=1}^{n} \nu(\Pi_{\{i\}}^{-1}(\vec{s}_i) \times \mathcal{X}_{\mathbb{N}} \,|\, \Pi_{[i-1]}^{-1}(\vec{s}_{[i-1]}) \times \mathcal{X}_{\mathbb{N}})
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{n} \mathbb{P}(Z_i = s_i | Z_{[i-1]} = \vec{s}_{[i-1]}) \\
&= \mathbb{P}(Z_{[n]} = \vec{s}_{[n]})
\end{aligned}
$$

and

$$
\begin{aligned}
&\nu(\mathcal{X}_{\mathbb{N}} \times \Pi_{[n]}^{-1}(\vec{s}_{[n]})) \\
&= \prod_{i=1}^{n} \nu(\mathcal{X}_{\mathbb{N}} \times \Pi_{\{i\}}^{-1}(s_i) \,|\, \mathcal{X}_{\mathbb{N}} \times \Pi_{[i-1]}^{-1}(\vec{s}_{[i-1]})) \\
&= \prod_{i=1}^{n} \left[(1-p_i)\, \mathbb{I}_{\{0\}}(s_i) + p_i\, \mathbb{I}_{\{1\}}(s_i)\right] \\
&= \mathbb{P}(X_{[n]} = \vec{s}_{[n]})\,.
\end{aligned}
$$

Hence the marginal of the first coordinate has the same law as Z and the marginal of the second coordinate has the law $\Pi_{\vec{p}}^{\mathbb{N}}$.

Finally we calculate (4.10c) for ν. We proceed by induction over n. The induction base is

$$
\nu(\{(\vec{s},\vec{t}) \in {\mathcal{X}_V}^2 : \vec{s}_1 \geq \vec{t}_1\}) = \nu(\{(\vec{s},\vec{t}) \in {\mathcal{X}_V}^2 : \vec{s}_1 = 0 < \vec{t}_1 = 1\}) = 0\,.
$$

The induction step is

$$
\begin{aligned}
&\nu(\{(\vec{s},\vec{t}) \in {\mathcal{X}_V}^2 : \vec{s}_{[n]} \geq \vec{t}_{[n]}\}) \\
&= \underbrace{\nu(\{(\vec{s},\vec{t}) \in {\mathcal{X}_V}^2 : \vec{s}_{[n-1]} \geq \vec{t}_{[n-1]}\})}_{=1 \text{ by induction}} \\
&\times \left(1 - \underbrace{\nu(\{(\vec{s},\vec{t}) \in {\mathcal{X}_V}^2 : \vec{s}_n = 0 < \vec{t}_n = 1 \,|\, \vec{s}_{[n-1]} \geq \vec{t}_{[n-1]}\})}_{=0 \text{ by definition of } \nu}\right) \\
&= 1\,.
\end{aligned}
$$

Hence

$$
\forall\, n \in \mathbb{N} : \quad \nu(\{(\vec{s},\vec{t}) \in {\mathcal{X}_V}^2 : \vec{s}_{[n]} \ngeq \vec{t}_{[n]}\}) = 0\,.
$$

This implies that

$$
\nu(\{(\vec{s},\vec{t}) \in {\mathcal{X}_V}^2 : \vec{s} \ngeq \vec{t}\}) = 0\,.
$$

□

Proposition 96. *Let Y and Z be two BRFs over the same index space V. Then*

$$
Y \wedge Z \overset{st}{\leq} Y \overset{st}{\leq} Y \vee Z\,. \tag{4.8a}
$$

If X is a third BRF independent of (Y, Z), then also

$$
Y \overset{st}{\geq} Z \Rightarrow \begin{cases} (Y \wedge X) \overset{st}{\geq} (Z \wedge X)\,, \\ (Y \vee X) \overset{st}{\geq} (Z \vee X)\,. \end{cases} \tag{4.8b}
$$

Proof. (4.8a): Take a finite $W \subseteq V$ and $f \in \mathrm{Mon}(W)$. Then

$$\begin{aligned}
&\mathbb{E}[f(Y_W \wedge Z_W)]\\
&= \sum_{\vec{z}\in\operatorname{supp} Z_W} \mathbb{E}[f(Y_W \wedge \vec{z})|Z_W = \vec{z}]\mathbb{P}(Z_W = \vec{z})\\
&\leq \sum_{\vec{z}\in\operatorname{supp} Z_W} \mathbb{E}[f(Y_W)|Z_W = \vec{z}]\mathbb{P}(Z_W = \vec{z})\\
&= \mathbb{E}[f(Y_W)]\\
&= \sum_{\vec{z}\in\operatorname{supp} Z_W} \mathbb{E}[f(Y_W)|Z_W = \vec{z}]\mathbb{P}(Z_W = \vec{z})\\
&\leq \sum_{\vec{z}\in\operatorname{supp} Z_W} \mathbb{E}[f(Y_W \vee \vec{z})|Z_W = \vec{z}]\mathbb{P}(Z_W = \vec{z})\\
&= \mathbb{E}[f(Y_W \vee Z_W)]\,.
\end{aligned}$$

Hence $Y_W \wedge Z_W \overset{st}{\leq} Y_W \overset{st}{\leq} Y_W \vee Z_W$. Conclude with lemma 93.

(4.8b): Take a finite $W \subseteq V$ and $f \in \mathrm{Mon}(W)$. For $\vec{x} \in \mathcal{X}_W$ and $f \in \mathrm{Mon}(W)$ define

$$f_{\vec{x}} : \quad \mathcal{X}_W \to \mathbb{R} \qquad \vec{y} \mapsto f(\vec{y} \wedge \vec{x})\,.$$

Then $f_{\vec{x}} \in \mathrm{Mon}(W)$, as

$$\vec{y} \leq \vec{z} \Rightarrow \vec{y} \vee \vec{x} \leq \vec{z} \vee \vec{x} \Rightarrow f_{\vec{x}}(\vec{y}) = f(\vec{y} \vee \vec{x}) \leq f(\vec{z} \vee \vec{x}) = f_{\vec{z}}(\vec{y})\,.$$

We get

$$\begin{aligned}
&\mathbb{E}[f(Y_W \vee X_W)]\\
&= \sum_{\vec{x}\in\mathcal{X}_W} \mathbb{E}[f(Y_W \wedge \vec{x})]\mathbb{P}(X_W = \vec{x})\\
&= \sum_{\vec{x}\in\mathcal{X}_W} \mathbb{E}[f_{\vec{x}}(Y_W)]\mathbb{P}(X_W = \vec{x})\\
&\geq \sum_{\vec{x}\in\mathcal{X}_W} \mathbb{E}[f_{\vec{x}}(Z_W)]\mathbb{P}(X_W = \vec{x}) \qquad \text{as } Y_W \overset{st}{\geq} Z_W \text{ and } f \in \mathrm{Mon}(W)\\
&= \mathbb{E}[f(Z_W \vee X_W)]\,.
\end{aligned}$$

The same derivation holds for $\wedge$ instead of $\vee$. The fact that X is independent of (Y, Z) is crucial, as we do not know if $(Y_W|X = \vec{x}) \overset{st}{\geq} (Z_W|X = \vec{x})$. Conclude with lemma 93. □

Proposition 97. *Let C be one of the dependency classes used in this paper. Then for all $\vec{p}$ and $\vec{r}$ we have*

$$\Sigma(C(\vec{p}\,\vec{r})) \subseteq \Sigma(C(\vec{p}))\,. \tag{4.9}$$

Proof. Let $\vec{c} \in \Sigma(C(\vec{p}\,\vec{r}))$. Let $Y \in C(\vec{p})$ and X be $\Pi^V_{\vec{r}}$-distributed independently of Y. Using (4.8a) we get $\Pi^V_{\vec{c}} \overset{st}{\leq} Y \wedge X \overset{st}{\leq} Y$, whence $\vec{c} \in \Sigma(Y)$. As this holds for every $Y \in C(\vec{p})$ we have $\vec{c} \in \Sigma(C(\vec{p}))$. □

Lemma 98 ([Lig05, chapter II, theorem 2.4])**.** *Let Y, Z be two BRFs indexed by V, then $Y \overset{st}{\geq} Z$ iff there exists a $\nu \in \mathcal{M}_1({\mathcal{X}_V}^2)$ such that*

$$\forall \text{ finite } W \subseteq V, \forall \vec{s}_W \in \mathcal{X}_W : \quad \nu(\Pi_W^{-1}(\vec{s}_W) \times \mathcal{X}_V) = \mathbb{P}(Y_W = \vec{s}_W) \tag{4.10a}$$

$$\forall \text{ finite } W \subseteq V, \forall \vec{t}_W \in \mathcal{X}_W : \quad \nu(\mathcal{X}_V \times \Pi_W^{-1}(\vec{t}_W)) = \mathbb{P}(Z_W = \vec{t}_W) \tag{4.10b}$$

$$\nu(\{(\vec{s}, \vec{t}) \in {\mathcal{X}_V}^2 : \vec{s} \geq \vec{t}\}) = 1\,. \tag{4.10c}$$

Remark. The coupling probability measure ν in lemma 98 is in general not unique.

Proposition 99. *Let Y and Z be two BRFs indexed by the same set V. Then we have:*

$$Y \overset{st}{\geq} Z \Rightarrow \forall \text{ finite } W \subseteq V : \quad \begin{pmatrix} \mathbb{P}(Y_W = \vec{1}) \geq \mathbb{P}(Z_W = \vec{1}) \\ \textit{and} \\ \mathbb{P}(Y_W = \vec{0}) \leq \mathbb{P}(Z_W = \vec{0}) \end{pmatrix}. \tag{4.11}$$

Proof. Assume that $Y \overset{st}{\geq} Z$ and let $W \subseteq V$ be finite. Lemma 93 asserts that $Y_W \overset{st}{\geq} Z_W$. Regard the monotone functions $f = \mathbb{I}_{\Pi_W^{-1}(\vec{1})}$ and $g = 1 - \mathbb{I}_{\Pi_W^{-1}(\vec{0})}$. Stochastic domination implies that

$$\mathbb{P}(Y_W = \vec{1}) = \mathbb{E}[f(Y)] \geq \mathbb{E}[f(Z)] = \mathbb{P}(Z_W = \vec{1})$$

and

$$\mathbb{P}(Y_W = \vec{0}) = 1 - \mathbb{E}[g(Y)] \leq 1 - \mathbb{E}[g(Z)] = \mathbb{P}(Z_W = \vec{0})\,.$$

□

Proposition 100. *Let Y be a BRF taking values in $\mathcal{X}_V$. Then $\Sigma(Y)$ is closed and a down-set.*

Proof. Take a finite $W \subseteq V$. Then $\Sigma(Y_W)$ is closed because we have a finite number of inequalities over the space of probability measures on $\mathcal{X}_W$, which is at most $2^{|W|}$-dimensional. If $\vec{c} \in \Sigma(Y_W)$ and $\vec{d} \leq \vec{c}$, then $\Pi_{\vec{d}}^W \overset{st}{\leq} \Pi_{\vec{c}}^W \overset{st}{\leq} Y$. Therefore $\Sigma(Y_W)$ is a down-set. Those properties then carry over to $\Sigma(Y)$ by taking the limit in the net of finite subsets of V. □

4.4.2 Nondomination

In this section we prove inclusion (ND) from figure 4.1, that is $\mathcal{P}_{dom}^{\mathcal{C}_G^{\text{strong}}} \subseteq \mathring{\mathcal{P}}_{sh}^G$. The plan is as follows: in lemma 101 we recall a coupling involving Shearer's measure on a finite graph H [She85, proof of theorem 1], which creates a BRF dominating only trivial BPFs for every $\vec{p} \notin \mathring{\mathcal{P}}_{sh}^H$. In proposition 102 we generalize an approach used by Liggett, Schonmann & Stacey [LSS97, theorem 2.1] to arbitrary graphs and inhomogeneous parameters. For infinite G and $\vec{p} \notin \mathring{\mathcal{P}}_{sh}^G$ we find a suitable finite subgraph H of G on which to effectuate the above mentioned coupling and extend it with an independent BPF on the complement. The resulting BRF dominates only trivial BPFs.

Lemma 101 ([She85, proof of theorem 1])**.** *Let G be finite. If $\vec{p} \notin \mathring{\mathcal{P}}_{sh}^G$, then there exists a BRF $Z \in \mathcal{C}_G^{strong}(\vec{p})$ with $\mathbb{P}(Z_V = \vec{1}) = 0$.*

Proof. As $\vec{p} \notin \mathring{\mathcal{P}}_{sh}^{G}$ and $\vec{1} \in \mathring{\mathcal{P}}_{sh}^{G}$ the line segment $[\vec{p}, \vec{1}]$ crosses $\partial\mathcal{P}_{sh}^{G}$ at the vector $\vec{r}$ (unique because $\mathring{\mathcal{P}}_{sh}^{G}$ is an up-set [SS05, proposition 2.15 (b)]). Let $\vec{x}$ be the solution of $\vec{p} = \vec{x}\,\vec{r}$. Let Y be $\mu_{G,\vec{r}}$-distributed and X be $\Pi_{\vec{x}}^{V}$-distributed independently of Y. Set $Z := Y \wedge X$. Then $Z \in \mathcal{C}_G^{\text{strong}}(\vec{p})$ and

$$\mathbb{P}(Z_V = \vec{1}) = \mathbb{P}(X_V = \vec{1})\mu_{G,\vec{r}}(Y_V = \vec{1}) = 0\,.$$

□

Proposition 102. *We have* $\mathcal{P}_{dom}^{\mathcal{C}_G^{strong}} \subseteq \mathring{\mathcal{P}}_{sh}^{G}$.

Proof. Let $\vec{p} \notin \mathring{\mathcal{P}}_{sh}^{G}$. Then there exists a finite set $W \subseteq V$ such that $\vec{p}_W \notin \mathring{\mathcal{P}}_{sh}^{G(W)}$. Using lemma 101 create a $Y_W \in \mathcal{C}_{G(W)}^{\text{strong}}(\vec{p})$ with $\mathbb{P}(Y_W = \vec{1}) = 0$. Extend this to a $Y \in \mathcal{C}_G^{\text{strong}}(\vec{p})$ by letting $Y_{V\setminus W}$ be $\Pi_{\vec{p}_{V\setminus W}}^{V\setminus W}$-distributed independently of Y_W. Suppose that $Y \overset{st}{\geq} X$, where X is $\Pi_{\vec{x}}^{V}$-distributed. Then lemma 93 implies that $Y_W \overset{st}{\geq} X_W$ and, using $f := \mathbb{I}_{\{\vec{1}\}} \in \text{Mon}(W)$, that

$$0 = \mathbb{P}(Y_W = \vec{1}) = \mathbb{E}[f(Y_W)] \geq \mathbb{E}[f(X_W)] = \mathbb{P}(X_W = \vec{1}) = \prod_{v \in W} x_v \geq 0\,.$$

Hence there exists a $v \in W$ such that $x_v = 0$, whence $\vec{x} \not> \vec{0}$ and $\vec{p} \notin \mathcal{P}_{dom}^{\mathcal{C}_G^{\text{strong}}}$. □

4.4.3 Domination

In this section we prove inclusion (UD) from figure 4.1, that is $\mathring{\mathcal{P}}_{sh}^{G} \subseteq \mathcal{P}_{udom}^{\mathcal{C}_G^{\text{weak}}}$. We split the proof in two and deal with finite and infinite G separately in proposition 103 and 104, respectively. Additionally (4.12) and (4.13) combined yield a proof of (4.4) from theorem 92.

On a finite graph our approach is direct: proposition 103 uses the minimality of $\mu_{G,\vec{p}}$ to construct a homogeneous nontrivial parameter vector $\vec{0} < \vec{c} \in \Sigma(\mathcal{C}_G^{\text{weak}}(\vec{p}))$. For an infinite graph the situation is more involved and we use a technique of Antal & Pisztora [AP96, pages 1040–1041]: Suppose you have a $Y \in \mathcal{C}_G^{\text{weak}}(\vec{p})$ with $\vec{0} < \vec{y} \in \Sigma(Y)$. Let X be $\Pi_V^{\vec{x}}$ with $\vec{0} < \vec{x}$ independently of Y and set $Z := X \wedge Y$. Then $\vec{0} < \vec{x}\,\vec{y} \in \Sigma(Z) \subsetneq \Sigma(Y)$, that is an independent non-trivial i.i.d. perturbation does not change the quality of Y's domination behaviour.

Proposition 105 uses this perturbation to blame adjacent 0 realizations of Z on X instead of Y, leading to the uniform technical minorization (4.14):

$$\mathbb{P}(Z_v = 1|Z_W = \vec{s}_W) \geq q_v \alpha_W^v(\vec{p})\,,$$

connecting the domination problem with Shearer's measure. Finally in proposition 104 we ensure to look at only escaping (W, v)s, hence getting rid of the $\alpha_W^v(\vec{p})$ term. This allows us to apply proposition 94, a variant of [Rus82, lemma 1], and guarantee stochastic domination of a non-trivial BPF.

Proposition 103. *Let G be finite and $\vec{p} \in \mathring{\mathcal{P}}^G_{sh}$. Let X be Π^V_c-distributed with*

$$c := 1 - (1 - \Xi_G(\vec{p}))^{1/|V|} > 0\,. \tag{4.12}$$

Then every $Y \in \mathcal{C}^{weak}_G(\vec{p})$ fulfils $Y \overset{st}{\geq} X$, hence $\vec{p} \in \mathcal{P}^{\mathcal{C}^{weak}_G}_{udom}$.

Proof. The choice of $\vec{p}$ implies that $\Xi_G(\vec{p}) > 0$, therefore $c > 0$, too. Let $f \in \mathrm{Mon}(V)$ and $Y \in \mathcal{C}^{\mathrm{weak}}_G(\vec{p})$. Then

$$\begin{aligned}
&\mathbb{E}[f(X)] \\
&= \sum_{\vec{s} \in \mathcal{X}_V} f(\vec{s})\,\mathbb{P}(X = \vec{s}) \\
&\leq f(\vec{0})\,\mathbb{P}(X = \vec{0}) + f(\vec{1})\,\mathbb{P}(X \neq \vec{0}) && \text{monotonicity of } f \\
&= f(\vec{0})(1-c)^{|V|} + f(\vec{1})[1 - (1-c)^{|V|}] \\
&= f(\vec{0})[1 - \Xi_G(\vec{p})] + f(\vec{1})\,\Xi_G(\vec{p}) \\
&\leq f(\vec{0})\,\mathbb{P}(Y \neq \vec{1}) + f(\vec{1})\,\mathbb{P}(Y = \vec{1}) && \text{minimality (2.28b)} \\
&\leq \sum_{\vec{s} \in \mathcal{X}_V} f(\vec{s})\,\mathbb{P}(Y = \vec{s}) && \text{monotonicity of } f \\
&= \mathbb{E}[f(Y)]\,.
\end{aligned}$$

Hence $X \overset{st}{\leq} Y$. As $\vec{0} < c\,\vec{1}$ we have $\vec{p} \in \mathcal{P}^{\mathcal{C}^{\mathrm{weak}}_G}_{udom}$. □

Proposition 104. *Let G be infinite and connected. Let $\vec{1} > \vec{p} \in \mathring{\mathcal{P}}^G_{sh}$. Define the vector $\vec{c}$ by*

$$\forall\, v \in V: \quad c_v := q_v \min\{q_w : w \in \mathcal{N}(v)\}\,. \tag{4.13}$$

Then $\vec{c} > \vec{0}$ and every $Y \in \mathcal{C}^{weak}_G(\vec{p})$ fulfils $Y \overset{st}{\geq} \Pi^V_{\vec{c}}$, whence $\vec{p} \in \mathcal{P}^{\mathcal{C}^{weak}_G}_{udom}$.

Remark. Proposition 104 motivated the definition of "escaping" pairs: it allows for non-trivial lower bounds for escaping $\alpha^v_W(\vec{p})$, in a correctly chosen ordering of a finite subgraph. Arbitrary $\alpha^v_W(\vec{p})$ defy control of this kind.

Proof. We show that $Y_W \overset{st}{\geq} \Pi^W_{\vec{c}_W}$ for every finite $W \subsetneq V$. Admitting this momentarily lemma 93 asserts that $Y \overset{st}{\geq} \Pi^V_{\vec{c}}$. Conclude as $\vec{p} < \vec{1}$ implies that $\vec{c} > \vec{0}$.

Choose a finite $W \subsetneq V$ and let $|W| =: n$. As G is connected and infinite there is a vertex $v_n \in W$ which has a neighbour w_n in $V \setminus W$. It follows $(W \setminus \{v_n\}, v_n)$ is escaping with escape $w_n \in \mathcal{N}(v_n) \setminus W$. Apply this argument recursively to $W \setminus \{v_n\}$ and thus produce a total ordering $v_1 < \ldots < v_n$ of W, where, setting $W_i := \{v_1, \ldots, v_{i-1}\}$, every (W_i, v_i) is escaping with escape $w_i \in \mathcal{N}(v_i) \setminus W_i$.

Let X be $\Pi^V_{\vec{q}}$-distributed independently of Y. Set $Z := Y \wedge X$. Then (4.14) from proposition 105 and the minoration for escaping pairs (2.25b) combine to

$$\forall\, i \in [n], \forall\, \vec{s}_{W_i} \in \mathcal{X}_{W_i}: \quad \mathbb{P}(Z_{v_i} = 1 | Z_{W_i} = \vec{s}_{W_i}) \geq \alpha^{v_i}_{W_i}(\vec{p}) q_{v_i} \geq q_{w_i} q_{v_i} \geq c_{v_i}\,.$$

This is sufficient for proposition 94 to construct a coupling with $Z_W \overset{st}{\geq} \Pi^W_{\vec{c}_W}$. Apply (4.8a) to get

$$Y_W \overset{st}{\geq} Y_W \wedge X_W = Z_W \overset{st}{\geq} \Pi^W_{\vec{c}_W}$$

and extend this to all of V with the help of lemma 93. □

Proposition 105. *Let $\vec{1} > \vec{p} \in \mathring{\mathcal{P}}^G_{sh}$ and $Y \in \mathcal{C}^{weak}_G(\vec{p})$. Let X be $\Pi^V_{\vec{q}}$-distributed independently of Y and set $Z := X \wedge Y$. We claim that for all admissible (W, v)*

$$\forall\, \vec{s}_W \in \mathcal{X}_W : \quad \mathbb{P}(Z_v = 1 | Z_W = \vec{s}_W) \geq q_v \alpha^v_W(\vec{p})\,. \tag{4.14}$$

Remark. This generalizes [LSS97, proposition 1.2], the core of Liggett, Schonmann & Stacey's proof, in the following ways: we localize the parameters α and r they used and assume no total ordering of the vertices yet. Furthermore $r_v = q_v$ follows from a conservative bound of the form

$$r_v := 1 - \sup\{\alpha^v_W(\vec{p}) : (W, v) \text{ escaping}\} = 1 - p_v = q_v\,,$$

where the sup is attained in $\alpha^v_\emptyset(\vec{p}) = p_v$.

Note to self: Lopsided conditions are sufficient for the LLL but fail in the proof of proposition 105 in (4.16c), as in general $\vec{s}_M \neq \vec{1}$. ♢

Proof. Recall that $\vec{p} \in \mathring{\mathcal{P}}^G_{sh}$ implies that $\vec{p} > \vec{0}$. Whence $\vec{q} < \vec{1}$ and (4.14) is well defined because

$$\forall \text{ finite } W \subseteq V, \vec{s}_W \in \mathcal{X}_W : \quad \mathbb{P}(Z_W = \vec{s}_W) > 0\,.$$

For every decomposition $N_0 \uplus N_1 := \mathcal{N}(v) \cap W$ with $N_0 =: \{u_1, \ldots, u_l\}$, $N_1 =: \{w_1, \ldots, w_m\}$ and $M := W \setminus \mathcal{N}(v)$ the fundamental identity (2.23) implies the inequality

$$[1 - \alpha^v_W(\vec{p})] \left(\prod_{j=1}^{l} p_{u_j}\right) \prod_{i=1}^{m} \alpha^{w_i}_{M \uplus \{w_1, \ldots, w_{i-1}\}}(\vec{p}) \geq q_v\,, \tag{4.15}$$

where $p_{u_j} \geq \alpha^{u_j}_{M \uplus N_1 \uplus \{u_1, \ldots, u_{j-1}\}}(\vec{p})$ follows from (2.25a).

We prove (4.14) inductively over the cardinality of W. The induction base $W = \emptyset$ is easy as $\mathbb{P}(Z_v = 1) = q_v \mathbb{P}(Y_v = 1) \geq q_v p_v = q_v \alpha^v_\emptyset(\vec{p})$. For the induction step fix $\vec{s}_W \in \mathcal{X}_W$ and the decomposition

$$N_0 := \{w \in W \cap \mathcal{N}(v) : s_w = 0\} =: \{u_1, \ldots, u_l\}$$

and

$$N_1 := \{w \in W \cap \mathcal{N}(v) : s_w = 1\} =: \{w_1, \ldots, w_m\}\,.$$

We write

$$\begin{aligned} &\mathbb{P}(Y_v = 0 | Z_W = \vec{s}_W) \\ &= \mathbb{P}(Y_v = 0 | Z_{N_0} = \vec{0}, Z_{N_1} = \vec{1}, Z_M = \vec{s}_M) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathbb{P}(Y_v = 0, Z_{N_0} = \vec{0}, Z_{N_1} = \vec{1}, Z_M = \vec{s}_M)}{\mathbb{P}(Z_{N_0} = \vec{0}, Z_{N_1} = \vec{1}, Z_M = \vec{s}_M)} \\
&\leq \frac{\mathbb{P}(Y_v = 0, Z_M = \vec{s}_M)}{\mathbb{P}(X_{N_0} = \vec{0}, Y_{N_1} = \vec{1}, Z_M = \vec{s}_M)} && (4.16a) \\
&= \frac{\mathbb{P}(Y_v = 0 | Z_M = \vec{s}_M)\mathbb{P}(Z_M = \vec{s}_M)}{\mathbb{P}(X_{N_0} = \vec{0})\,\mathbb{P}(Y_{N_1} = \vec{1}, Z_M = \vec{s}_M)} && (4.16b) \\
&\leq \frac{q_v}{\mathbb{P}(X_{N_0} = \vec{0})\,\mathbb{P}(Y_{N_1} = \vec{1} | Z_M = \vec{s}_M)} && (4.16c) \\
&= \frac{q_v}{\prod_{j=1}^{l}(1 - q_{u_j}) \prod_{i=1}^{m} \mathbb{P}(Y_{w_i} = 1 | Y_{w_1} = \ldots = Y_{w_{i-1}} = 1, Z_M = \vec{s}_M)} \\
&\leq \frac{q_v}{\prod_{j=1}^{l} p_{u_j} \prod_{i=1}^{m} \alpha^{w_i}_{M \uplus \{w_1, \ldots, w_{i-1}\}}(\vec{p})} && (4.16d) \\
&\leq 1 - \alpha^v_W(\vec{p})\,. && (4.16e)
\end{aligned}
$$

The key steps in (4.16) are:

(4.16a) increasing the numerator by dropping $Z_{N_0} = \vec{0}$ and $Z_{N_1} = \vec{1}$ while decreasing the denominator by using the definition of Z,

(4.16c) as $d(v, M) \geq 1$ and $Y \in \mathcal{C}^{\text{weak}}_G(\vec{p})$,

(4.16b) using the independence of X_{N_0} from (Y_{N_1}, Z_M),

(4.16d) applying the induction hypothesis (4.14) to the factors of the rhs product in the denominator, which have strictly smaller cardinality,

(4.16e) applying inequality (4.15).

Hence

$$
\mathbb{P}(Z_v = 1 | Z_W = \vec{s}_W) \geq q_v \mathbb{P}(Y_v = 1 | Z_W = \vec{s}_W) \geq q_v \alpha^v_W(\vec{p})\,.
$$

□

4.5 The weak invariant case

In this section we extend our characterization to the case of BRFs with weak dependency graph which are invariant under a group action. Let Γ be a subgroup of $\text{Aut}(G)$. A BRF Y is Γ-*invariant* iff

$$
\forall\, \gamma \in \Gamma : \quad (\gamma Y) := (Y_{\gamma(v)})_{v \in V} \text{ has the same law as } Y\,. \tag{4.17}
$$

For a given Γ and Γ-invariant $\vec{p}$ we denote by $\mathcal{C}^{\text{weak}}_{\Gamma\text{-inv}}(\vec{p})$ the *weak,* Γ*-invariant dependency class*, that is Γ-invariant BRFs with weak dependency graph G, and by $\mathcal{C}^{\text{strong}}_{\Gamma\text{-inv}}(\vec{p})$ the corresponding strong version.

We call a pair (G, Γ) *partition exhaustive* iff there exists a sequence of partitions $(P_n)_{n \in \mathbb{N}}$ of V with $P_n := (V^n_i)_{i \in \mathbb{N}}$ such that

$$
\forall\, n, i, j \in \mathbb{N} : \quad G(V^{(n)}_i) \text{ is isomorph to } G(V^{(n)}_1) =: G_n\,, \tag{4.18a}
$$

$$\forall\, n \in \mathbb{N}: \quad \text{the orbit of } P_n \text{ under } \Gamma \text{ is finite,} \tag{4.18b}$$

$$V_1^{(n)} \xrightarrow[n\to\infty]{} V, \text{ that is } (G_n)_{n\in\mathbb{N}} \text{ exhausts } G. \tag{4.18c}$$

The kind of graphs we have in mind are regular infinite trees and tree-like graphs, $\mathbb{Z}^d$ and other regular lattices (triangular, hexagonal, ...). We think of the group Γ to be generated by some of the natural shifts and rotations of the graph. An example for a sequence of partitions would be increasing regular rectangular decompositions of $\mathbb{Z}^d$.

Theorem 106. *Let (G, Γ) be partition exhaustive. Then*

$$\mathcal{P}_{udom}^{\mathcal{C}_{\Gamma\text{-}inv}^{weak}} = \mathcal{P}_{dom}^{\mathcal{C}_{\Gamma\text{-}inv}^{weak}} = \mathring{\mathcal{P}}_{sh}^{\Gamma\text{-}inv} := \{\vec{p} \in \mathring{\mathcal{P}}_{sh}^{G} : \vec{p} \text{ is } \Gamma\text{-invariant}\}\,. \tag{4.19}$$

Remark. It follows from (4.18) that Γ acts quasi-transitively on G. Hence $\mathring{\mathcal{P}}_{sh}^{\Gamma\text{-inv}}$ can be seen as a subset of a finite-dimensional space.

Proof. As $\mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}(\vec{p})$ is a subclass of $\mathcal{C}_{G}^{\text{weak}}(\vec{p})$ theorem (91) implies that $\mathring{\mathcal{P}}_{sh}^{\Gamma\text{-inv}} \subseteq \mathcal{P}_{udom}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}} \subseteq \mathcal{P}_{dom}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}}$. We show $\mathcal{P}_{dom}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}} \subseteq \mathring{\mathcal{P}}_{sh}^{\Gamma\text{-inv}}$ by constructing a counterexample. Let $\vec{p} \notin \mathring{\mathcal{P}}_{sh}^{\Gamma\text{-inv}}$, then by (4.18c) there exists a $n \in \mathbb{N}$ such that $\vec{p} \notin \mathring{\mathcal{P}}_{sh}^{G_n,\Gamma}$ (the intersection of the projections of Γ-invariant parameters on G with $\mathring{\mathcal{P}}_{sh}^{G_n}$). Let $P := P_n$ and let $(P^{(1)}, \dots, P^{(k)})$ be its finite orbit under the action of Γ (4.18b). By (4.18a) each class $V_{(i,j)} \in P^{(j)}$ has a graph $G(V_{(i,j)})$ isomorph to G_n. Use lemma 101 to construct i.i.d. BPFs $Z^{(i,j)} \in \mathcal{C}_{G_n}^{\text{strong}}(\vec{p})$ with $\mathbb{P}(Z^{(i,j)} = \vec{1}) = 0$. For $j \in [k]$ collate the $Z_{(i,j)}$ to a BPF $Z^{(j)}$, this works as $P^{(j)}$ is a partition of G. By definition $Z^{(j)} \in \mathcal{C}_{\Gamma\text{-inv}}^{\text{strong}}(\vec{p})$. Finally let U be Uniform$([k])$-distributed of everything else. Define the final BPF Z by

$$Z := \sum_{j=1}^{k} [U = j] Z^{(j)}\,. \tag{4.20}$$

We claim that $Z \in \mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}(\vec{p})$. The mixing in (4.20) keeps $Z \in \mathcal{C}_{G}^{\text{weak}}(\vec{p})$. To see its Γ-invariance let $\gamma \in \Gamma$. The automorphism γ acts injectively on $(P^{(1)}, \dots, P^{(k)})$ and thus also on $[k]$. Therefore, using the fact that U is uniform and everything is constructed independently, we have

$$\gamma Z = \sum_{j=1}^{k} [U = j] \gamma Z^{(j)} = \sum_{j=1}^{k} [U = \gamma^{-1} j] Z^{(j)} = \sum_{j=1}^{k} [U = j] Z^{(j)} = Z\,.$$

□

4.5.1 Remarks about the strong invariant case

The mixing in (4.20) destroys strong independence even in simple cases as $G := \mathbb{Z}$ and Γ the group of shifts on $\mathbb{Z}$ [LSS97, end of section 2]. A minimal example for this is the following: let $G := (\{v, w\}, \emptyset)$, $X^{(1)}, X^{(2)} \in \mathcal{C}_{G}^{\text{strong}}(\vec{p})$ and Y be Bernoulli($\frac{1}{2}$)-distributed, all independent of each other. Define $Z := X^{(Y)}$ and ask if $\mathbb{P}(Z_v = Z_w = 1) = \mathbb{P}(Z_v = 1)\mathbb{P}(Z_w = 1)$. This fails for most choices of $\vec{p}$. Calculations on slightly more complex graphs as $G := (\{u, v, w\}, \{(u, v)\})$ show that Z from (4.20) has no strong dependency graph. Thus the mixing approach, inspired by [LSS97, page 89], does not allow to characterize $\mathcal{P}_{udom}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{strong}}}$ and $\mathcal{P}_{dom}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{strong}}}$.

Coming back to the case $G := \mathbb{Z}$ and Γ the group of shifts on $\mathbb{Z}$ and the problem of determining $p_{udom}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{strong}}}$ we know that $p_{udom}^{\mathcal{C}_{\Gamma\text{-inv}}^{\text{strong}}} \in [\frac{1}{2}, \frac{3}{4}]$. Similar bounds are easy to show for $\mathbb{Z}_{(k)}$. The boundaries of the interval are given by known exact solutions of natural sub- and superclasses (in this case 2-factors and $\mathcal{C}_{\Gamma\text{-inv}}^{\text{weak}}$). A question is, whether the critical parameter coincides with one of the boundaries or lies strictly in the interval's interior. This goes hand in hand with an investigation in the discovery of *structural properties* of the above class. On the way towards this goal there are also some natural subclasses, like $(k+1)$-factors or Markovian, k-independent, translation-invariant BRFs, interesting in their own right. The use of mixing techniques, which solve the weak case [LSS97, Tem12], fails here. Possible approaches include applying dynamic systems techniques à la Aaronson et al. [AGKdV89], factorizing the probability measures [Bro64] or specializing more general representation theorems [Ros59, Ros63, Ros10].

We know, that if a BRF $Y := (Y_v)_{v\in V}$ stochastically dominates a BPF $X := (X_v)_{v\in V}$ with parameter $\vec{c}$, then

$$\forall\, W \Subset V: \quad \mathbb{P}(Y_W = \vec{1}) \geq \mathbb{P}(X_W = \vec{1}) = \prod_{v\in W} c_v\,.$$

One may ask, if the converse also holds. That is

Question 107. Let $Y := (Y_v)_{v\in V}$ be a BRF and $\vec{c} \in]0,1]^V$. Suppose that

$$\forall\, W \Subset V: \quad \mathbb{P}(Y_W = \vec{1}) \geq \mathbb{P}(X_W = \vec{1}) = \prod_{v\in W} c_v\,. \tag{4.21}$$

Does there exist a vector $\vec{x} \in]0,1]^V$, such that $Y \overset{st}{\geq} \Pi_V^{\vec{x}}$.

Assuming that this question can be answered positively, another approach to $p_{udom}^{\mathcal{C}_{\text{-inv}}^{\text{strong}}\Gamma}$, for given G, might be to look, if one can use invariance and strong independence to get conditions on $\vec{p}$ to have (4.21).

4.6 The asymptotic size of the jump on $\mathbb{Z}_{(k)}$

Liggett, Schonmann & Stacey stated the following conjecture about the size of the jump at the critical value:

Conjecture 108 ([LSS97, after corollary 2.2])**.**

$$\forall\, k \in \mathbb{N}_0: \quad \sigma(\mathcal{C}_{\mathbb{Z}_{(k)}}^{weak}(p_{udom}^{\mathcal{C}_{\mathbb{Z}_{(k)}}^{weak}})) = \frac{k}{k+1}\,. \tag{4.22}$$

We think that Liggett, Schonmann & Stacey were motivated by the fact that they obtained the lower bound using some extra randomness (see the Y in [LSS97, proposition 1.2] or the X in the proof of proposition 105). Without this extra randomness we would have $\sigma(\mu_{G,p}) = \sigma(\mathcal{C}_G^{\text{weak}}(p_{udom}^{\mathcal{C}_G^{\text{weak}}}))$. On

$\mathbb{Z}_{(k)}$, the k-fuzz of $\mathbb{Z}$, we have $p_{sh}^{\mathbb{Z}_{(k)}} = 1 - \frac{k^k}{(k+1)^{(k+1)}}$ [MT12, section 4.2] and $\sigma(\mu_{\mathbb{Z}_{(k)}, p_{sh}^{\mathbb{Z}_{(k)}}}) = \frac{k}{k+1}$ [MT12, section 4.2]. We show that this intuition is wrong and that as $k \to \infty$ the dependence ranges further along $\mathbb{Z}$, making the effect of adding randomness second to it. In proposition 109 we show that asymptotically $\sigma(\mathcal{C}^{\text{weak}}_{\mathbb{Z}_{(k)}}(p_{sh}^{\mathbb{Z}_{(k)}}))$ is much closer to the lower bound of $\frac{k}{(k+1)^2}$ from [LSS97, corollary 2.5].

Proposition 109. *We have*

$$\forall \varepsilon > 0 : \exists K(\varepsilon) : \forall k \geq K : \quad \sigma(\mathcal{C}^{weak}_{\mathbb{Z}_{(k)}}(p_{dom}^{\mathbb{Z}_{(k)}})) \leq \frac{1 + (1+\varepsilon)\ln(k+1)}{k+1} \; . \tag{4.23}$$

Proof. Let $\mathcal{N}_1(0)^+ := \{0, \dots, k\}$ be the nonnegative closed halfball of radius k centred at 0. Define a BRF Y on $\mathbb{Z}$ by setting $\mathbb{P}(Y_{\mathcal{N}_1(0)^+} = \vec{1}) := p_{dom}^{\mathbb{Z}_{(k)}}$, $\mathbb{P}(Y_{\mathcal{N}_1(0)^+} = \vec{0}) := q_{dom}^{\mathbb{Z}_{(k)}}$ and letting $Y_{\mathbb{Z} \setminus \mathcal{N}_1(0)^+}$ be $\Pi^{\mathbb{Z} \setminus \mathcal{N}_1(0)^+}_{p_{dom}^{\mathbb{Z}_{(k)}}}$-distributed independently of $Y_{\mathcal{N}_1(0)^+}$. As $Y \in \mathcal{C}^{\text{weak}}_{\mathbb{Z}_{(k)}}(p_{dom}^{\mathbb{Z}_{(k)}})$ [LSS97, corollary 2.5] applies and $Y \overset{st}{\geq} X$, where X is $\Pi^{\mathbb{Z}}_{\sigma}$-distributed with $\sigma \in [\frac{k}{(k+1)^2}, \frac{k}{k+1}]$. Lemma 93 implies $X_{\mathcal{N}_1(\vec{0})^+} \overset{st}{\leq} Y_{\mathcal{N}_1(\vec{0})^+}$ and in particular the inequality

$$1 - (1-\sigma)^{(k+1)} = \mathbb{P}(X_{\mathcal{N}_1(\vec{0})^+} \neq \vec{0}) = \mathbb{P}(Y_{\mathcal{N}_1(\vec{0})^+} \neq \vec{0}) = 1 - \frac{k^k}{(k+1)^{(k+1)}} \, .$$

Rewrite it into

$$\begin{aligned} \sigma &\leq 1 - \frac{k^{\frac{k}{k+1}}}{k+1} \\ &= \frac{1}{k+1} + \frac{k}{k+1}(1 - k^{-\frac{1}{k+1}}) \\ &\leq \frac{1}{k+1} + (1 - (k+1)^{-\frac{1}{k+1}}) \, . \end{aligned}$$

For every $\varepsilon > 0$ and z close enough to 0 we know that $1 - e^{-z} \leq (1+\varepsilon)z$. We conclude for $z_k := \frac{\ln(k+1)}{k+1} \xrightarrow[k\to\infty]{} 0$. □

4.7 Intrinsic stochastic domination for Shearer's measure

In this section we present two examples of stochastic dominations involving Shearer's measure. The first one results from the coupling in model 22 and relates Shearer's measure with different parameter vectors.

Proposition 110. *Let $\mathcal{P}^G_{sh} \ni \vec{p} \leq \vec{r}$. Then*

$$\mu_{\vec{p},G} \overset{st}{\leq} \mu_{\vec{r},G} \, . \tag{4.24}$$

Proof. By model (22) we can construct BRFs Y $\mu_{\vec{p},G}$-distributed and Z $\mu_{\vec{r},G}$-distributed, such that they are coupled explicitly by a BRF X and the relation $Z := Y \vee X$. Using (4.8a), we get $Z \overset{st}{\geq} Y$. □

The second one presents a BPF stochastically dominated by Shearer's measure. We find that the OVOEPs are a natural choice for the parameters of the dominated BRF. The procedure is a specialisation of chapter 4 to Shearer's measure. We specialize proposition 105 in proposition 111 and find that we do not need an auxiliary BPF. In special cases, that is on certain transitive graphs with homogeneous parameter, we find that this parametrization is optimal. See the examples in section 2.6.

Proposition 111. *Let $G := (V, E)$ be connected, $\vec{p} \in \mathring{\mathcal{P}}^G_{sh}$ and Y be $\mu_{G,\vec{p}}$-distributed. We claim, that*

$$\forall\, \vec{s}_W \in \mathcal{X}_W, (W, v): \quad \mu_{G,\vec{p}}(Y_v = 1 | Y_W = \vec{s}_W) \geq \alpha^v_W(\vec{p}) > 0\,. \tag{4.25}$$

Proof. The fact that $\vec{p} \in \mathring{\mathcal{P}}^G_{sh}$ implies that all admissible $\alpha^v_W(\vec{p})$ are well defined and non-zero.

We prove (4.25) inductively over the cardinality of W. The induction base for $W = \emptyset$ is $\mu_{G,\vec{p}}(Y_v = 1) = p_v = \alpha^v_\emptyset(\vec{p})$. For the induction step let $M := W \setminus \mathcal{N}(v)$ and $N := W \cap \mathcal{N}(v)$. Let $\vec{s}_W \in \mathcal{X}_W$ and assume that $\mu_{G,\vec{p}}(Y_W = \vec{s}_W) > 0$. The first case is $\vec{s}_N \neq \vec{1}$, whereby

$$\mu_{G,\vec{p}}(Y_v = 0 | Y_W = \vec{s}_W) = \frac{\mu_{G,\vec{p}}(Y_v = 0, Y_N \neq \vec{1}, Y_M = \vec{s}_M)}{\mu_{G,\vec{p}}(Y_Y = \vec{s}_W)} = 0\,,$$

as there are neighbouring zeros in (Y_v, Y_N). The second case is $\vec{s}_N = \vec{1}$. Let $\{w_1, \ldots, w_m\} := N$. Use the fundamental identity (2.23) to get

$$\begin{aligned}
&\mu_{G,\vec{p}}(Y_v = 0 | Y_W = \vec{s}_W)\\
&= \frac{\mu_{G,\vec{p}}(Y_v = 0, Y_N = \vec{1}, Y_M = \vec{s}_M)}{\mu_{G,\vec{p}}(Y_N = \vec{1}, Y_M = \vec{s}_M)}\\
&= \frac{\mu_{G,\vec{p}}(Y_v = 0)\mu_{G,\vec{p}}(Y_M = \vec{s}_M)}{\mu_{G,\vec{p}}(Y_N = \vec{1}, Y_M = \vec{s}_M)}\\
&= \frac{q_v}{\mu_{G,\vec{p}}(Y_N = \vec{1} | Y_M = \vec{s}_M)}\\
&= \frac{q_v}{\prod_{i=1}^m \mu_{G,\vec{p}}(Y_{w_i} = 1 | Y_{\{w_1,\ldots,w_{i-1}\}} = \vec{1}, Y_M = \vec{s}_M)}\\
&\leq \frac{q_v}{\prod_{i=1}^m \alpha^{w_i}_{M \uplus \{w_1,\ldots,w_{i-1}\}}(\vec{p})}\\
&= 1 - \alpha^v_W(\vec{p})\,.
\end{aligned}$$

□

Proposition 112. *Let G be infinite and connected. Assume that $\vec{1} > \vec{p} \in \mathring{\mathcal{P}}^G_{sh}$. Define the vector $\vec{x}$ by*

$$\forall\, v \in V: \qquad x_v := \min\{\alpha^v_{V \setminus \{v,w\}}(\vec{p}): \quad w \in \mathcal{N}(v)\}\,. \tag{4.26a}$$

Then

$$\mu_{G,\vec{p}} \overset{st}{\geq} \Pi^V_{\vec{x}}\,, \qquad \text{that is } \vec{x} \in \Sigma(\mu_{G,\vec{p}})\,, \tag{4.26b}$$

and

$$\forall\, v \in V : \quad x_v \geq \min\{q_w : w \in \mathcal{N}(v)\} > 0\,. \tag{4.26c}$$

Remark. If $\vec{p} \not< \vec{1}$, then regard the graph $G(W)$ with $W := \{v : p_v < 1\}$. If $G(W)$ has finite connected components, then apply proposition 103 to these. Infinite connected components of $G(W)$ are the domain of proposition 112.

Proof. The proof is the same proof as the one for proposition 104, except that, instead of using the auxiliary BPF X and (4.14) from proposition 105, we directly use (4.25) from proposition 111. ☐

4.8 A summary of the homogeneous case

In the homogeneous case each of the sets defined in (4.2), after being identified with the respective cross-sections, reduces to a one-dimensional interval described by its non-trivial endpoint. The *dominated Bernoulli parameter value* (short: dominated value) of a BPF Y is

$$\sigma(Y) := \max\{c : Y \overset{st}{\geq} \Pi_c^V\}\,. \tag{4.27a}$$

Note to self: For finite W we have $\sigma(Y_W) = \max\{c : Y_W \overset{st}{\geq} \Pi_c^W\}$. As $[0, \sigma(Y)] = \bigcap_{\text{finite } W \subseteq V} [0, \sigma(Y_W)]$ the value $\sigma(Y)$ is contained in all of them and a max. ♢

For a non-empty class C of BRFs this extends to

$$\sigma(C) := \inf\{\sigma(Y) : Y \in C\}\,. \tag{4.27b}$$

Note to self: Whereas in (4.27b) I have an inf instead of a min, because I can not guarantee that this value is attained for some BRF Y. ♢

The *critical domination values* of a class C, assuming that $C(p)$ is non-empty for all p, are written as

$$p^C_{dom} := \inf\{p \in [0,1] : \forall\, Y \in C(p) : \sigma(Y) > 0\} \tag{4.27c}$$

and

$$p^C_{udom} := \inf\{p \in [0,1] : \sigma(C(p)) > 0\}\,. \tag{4.27d}$$

As the function $p \mapsto \sigma(C(p))$ is non-decreasing (4.9) the sets $]p^C_{dom}, 1]$ and $]p^C_{udom}, 1]$ are up-sets and we have the inequality

$$p^C_{dom} \leq p^C_{udom}\,. \tag{4.27e}$$

The first known result is a bound on $p^{C^{\text{weak}}_G}_{udom}$ in the homogeneous case, only depending on the maximal degree of G:

Theorem 113 ([LSS97, theorem 1.3])**.** *If G has uniformly bounded degree by a constant D, then*

$$p^{C^{weak}_G}_{udom} \leq 1 - \frac{(D-1)^{(D-1)}}{D^D} \tag{4.28a}$$

and for $p \geq 1 - \frac{(D-1)^{(D-1)}}{D^D}$ *the dominated parameter is uniformly minorated:*

$$\sigma(\mathcal{C}_G^{weak}(p)) \geq \left(1 - \left(\frac{q}{(D-1)^{(D-1)}}\right)^{1/D}\right)\left(1 - (q(D-1))^{1/D}\right) . \tag{4.28b}$$

Additionally

$$\lim_{p \to 1} \sigma(\mathcal{C}_G^{weak}(p)) = 1 . \tag{4.28c}$$

Recall that for $k \in \mathbb{N}_0$ the *k-fuzz of* $G = (V, E)$ is the graph with vertices V and an edge for every pair of vertices at distance less than or equal to k in G. Denote the k-fuzz of $\mathbb{Z}$ by $\mathbb{Z}_{(k)}$. Note that $\mathbb{Z}_{(k)}$ is $2k$-regular. As $\mathbb{Z}_{(k)}$ has a natural order inherited from $\mathbb{Z}$ theorem 113 can be improved considerably:

Theorem 114 ([LSS97, theorems 0.0, 1.5 and corollary 2.2])**.** *On* $\mathbb{Z}_{(k)}$ *we have*

$$p_{dom}^{\mathcal{C}_{\mathbb{Z}_{(k)}}^{weak}} = p_{udom}^{\mathcal{C}_{\mathbb{Z}_{(k)}}^{weak}} = p_{dom}^{\mathcal{C}_{\mathbb{Z}_{(k)}}^{strong}} = p_{udom}^{\mathcal{C}_{\mathbb{Z}_{(k)}}^{strong}} = 1 - \frac{k^k}{(k+1)^{(k+1)}} . \tag{4.29a}$$

For $p \geq p_{udom}^{\mathcal{C}_{\mathbb{Z}_{(k)}}^{strong}}$ *the dominated parameter is minorated by*

$$\sigma(\mathcal{C}_{\mathbb{Z}_{(k)}}^{weak}(p)) \geq \left(1 - \left(\frac{q}{k^k}\right)^{\frac{1}{k+1}}\right)\left(1 - (qk)^{\frac{1}{k+1}}\right) . \tag{4.29b}$$

This implies a jump of $\sigma(\mathcal{C}_{\mathbb{Z}_{(k)}}^{weak}(.))$ *at the critical value* $p_{udom}^{\mathcal{C}_{\mathbb{Z}_{(k)}}^{weak}}$*, namely*

$$\forall\, k \in \mathbb{N}_0 : \quad \frac{k}{(k+1)^2} \leq \sigma(\mathcal{C}_{\mathbb{Z}_{(k)}}^{weak}(p_{udom}^{\mathcal{C}_{\mathbb{Z}_{(k)}}^{weak}})) . \tag{4.29c}$$

To arrive at the equality in (4.29a) Liggett, Schonmann & Stacey derived a lower bound from a particular probability measure, called Shearer's measure (see chapter 2). Furthermore it allowed them to show that

$$\forall\, k \in \mathbb{N}_0 : \quad \sigma(\mathcal{C}_{\mathbb{Z}_{(k)}}^{\text{strong}}(p_{udom}^{\mathcal{C}_{\mathbb{Z}_{(k)}}^{\text{strong}}})) \leq \frac{k}{k+1} . \tag{4.30}$$

Thus our main result can be written as a corollary of theorems 91 and 92:

Theorem 115. *Let* G *be a locally finite and connected graph. Then*

$$p_{dom}^{\mathcal{C}_G^{weak}} = p_{udom}^{\mathcal{C}_G^{weak}} = p_{dom}^{\mathcal{C}_G^{strong}} = p_{udom}^{\mathcal{C}_G^{strong}} = p_{sh}^G . \tag{4.31a}$$

If G *contains at least one infinite connected component and has uniformly bounded degree, then*

$$\sigma(\mathcal{C}_G^{weak}(p_{udom}^{\mathcal{C}_G^{weak}})) \geq (q_{udom}^{\mathcal{C}_G^{weak}})^2 > 0 , \tag{4.31b}$$

whereas if G *is finite we have*

$$\sigma(\mathcal{C}_G^{weak}(p_{udom}^{\mathcal{C}_G^{weak}})) = 0 . \tag{4.31c}$$

$$\begin{array}{ccccccc} p_{sh}^{G} & \overset{(\mathrm{UD})}{\geq} & p_{udom}^{\mathcal{C}_G^{\mathrm{weak}}} & \geq & p_{dom}^{\mathcal{C}_G^{\mathrm{weak}}} & & \\ & & \mathrel{\rotatebox{90}{$\geq$}} & & \mathrel{\rotatebox{90}{$\geq$}} & & \\ & & p_{udom}^{\mathcal{C}_G^{\mathrm{strong}}} & \geq & p_{dom}^{\mathcal{C}_G^{\mathrm{strong}}} & \overset{(\mathrm{ND})}{\geq} & p_{sh}^{G} \end{array}$$

Figure 4.2: Inequalities in the proof of (4.31). The four centre inequalities follow straight from (4.27e) and (2.3c). The inequality (ND) is an adaption of the approach used for $\mathbb{Z}_{(k)}$ in [LSS97], while inequality (UD) is the novel interpretation of the optimal bounds of Shearer's measure.

The discontinuity described in (4.31b) also holds for the more esoteric case of graphs having no uniform bound on their degree. In this case $p_{udom}^{\mathcal{C}_G^{\mathrm{strong}}} = 1$ and $\sigma(\mathcal{C}_G^{\mathrm{weak}}(1)) = 1 > 0$. An explanation for this discontinous transition might come from statistical mechanics, via the connection with hard-core lattice gases made by Scott & Sokal [SS05]. It should be equivalent to the existence of a non-physical singularity of the entropy for negative real fugacities for all infinite connected lattices.

The graph $\mathbb{Z}_{(k)}$ turns out to be a rare example of an infinite graph where we can construct Shearer's measure explicitly, in this case as a $(k+1)$-factor [MT12, section 4.2]. A second case immediately deducible from previous work would be the D-regular tree $\mathbb{T}_D$, where

$$1 - \frac{(D-1)^{(D-1)}}{D^D} = p_{sh}^{\mathbb{T}_D} \leq p_{dom}^{\mathbb{T}_D} \leq 1 - \frac{(D-1)^{(D-1)}}{D^D}$$

by [She85, theorem 2] and theorem 113.

Chapter 5

Cluster expansion of hard-core lattice gases

This chapter discusses convergence conditions for cluster expansions of abstract polymer models. We introduce *polymer systems* and give an account of their properties, including *cluster expansion*, *Mayer coefficients* and worst case behaviour in section 5.1. We present known conditions by Dobrushin and by Fernández & Procacci [FP07] in section 5.2.1. A detailed review of Fernández & Procacci's tree approximation techniques is given in section 5.5.

Our first contribution is a rigorous understanding of the relation between the location of the non-physical singularity and the domain of admissible parameters in section 5.2.3.

We also present a recent polymer level induction-based condition by Scott & Sokal [SS05] in 5.2.2. We build a theory of inductive partition schemes in section 5.4.4, combine them with Fernández & Procacci's techniques and derive an improved condition in section 5.6.

More inductive partition schemes, extensions and applications to other models are discussed in sections 5.7 and 6.3.3.

Notation 116. We denote vectors by $\vec{x}$ and all scalar operations/comparisons lift to vectors component-wise. Empty sums are 0 and empty products are 1. We interpret 0! as 1 and $A^0 = A^\emptyset = \emptyset$, for an arbitrary set A. We let $[n] := \{1, \dots, n\}$, $[n]_0 := \{0, 1, \dots, n\}$ and interpret $[0] := \emptyset$, while $[0]_0 := \{0\}$.

5.1 Introduction

5.1.1 Setup

Let $\mathcal{P}$ be a countable system of *polymers* equipped with a symmetric and reflexive *incompatibility relation* $\approx$. We assume that each polymer $\gamma \in \mathcal{P}$ is incompatible with only a finite number of other polymers, that is the graph $(\mathcal{P}, \approx)$ with vertex set $\mathcal{P}$ and edges given by $\approx$ is locally finite, does not have multiple edges and has a loop at every γ. Without loss of generality we additionally assume that the graph $(\mathcal{P}, \approx)$ is infinite and connected. Denote by $\mathcal{I}(\gamma) := \{\xi \in \mathcal{P} : \xi \approx \gamma\}$ and by $\mathcal{I}^\star(\gamma) := \mathcal{I}(\gamma) \setminus \{\gamma\}$. Recall that an *independent set* of polymers in $(\mathcal{P}, \approx)$ consists of *mutually compatible* polymers. Set $D_\gamma := |\mathcal{I}^\star(\gamma)|$, $D := \sup\{D_\gamma : \gamma \in \mathcal{P}\}$ and say that γ has *uniformly bounded incompatibilities* of degree D iff $D < \infty$. We denote the fact that a set of polymers Λ is a finite subset of $\mathcal{P}$ by $\Lambda \Subset \mathcal{P}$. For fixed $\Lambda \Subset \mathcal{P}$ we define the *grand canonical partition function* $\Xi_\Lambda : \mathbb{C}^\Lambda \to \mathbb{C}$ by

$$\Xi_\Lambda(\vec{z}) := \sum_{\text{compatible } I \subseteq \Lambda} \prod_{\gamma \in I} z_\gamma \tag{5.1a}$$

$$= \sum_{n \geq 0} \frac{1}{n!} \sum_{\vec{\xi} \in \Lambda^n} \left(\prod_{1 \leq i < j \leq n} [\xi_i \not\approx \xi_j] \right) \prod_{i=1}^{n} z_{\xi_i} \,. \tag{5.1b}$$

where $\vec{z}$ are the *activities* or *fugacities* on Λ. It follows that $\Xi_\emptyset(\vec{z}) = 1$. $\Xi_\Lambda(\vec{z})$ is affine with respect to each parameter z_γ.

Remark. Equation (5.1a) describes the *multi-index form* [Far10, (2.14)] of the Ξ_Λ. In this particular case the multi-indices degenerate into indicators of independent sets. Equation (5.1b) describes the *tensor form* [Far10, (2.13)] of the partition function. In this case Ξ_Λ corresponds to a *weighted (or multi-variable) exponential generating function.*

Finally we write $\Lambda \nearrow \mathcal{P}$ for taking the limit of an *exhausting sequence of finite subsets* (in the graph-theoretic sense) of $\mathcal{P}$.

5.1.2 The aim

There are two principal quantities of interest. From here on we assume that $\gamma \in \Lambda$. The first is the *reduced correlation* [GK71], which can always be written as a product of inverses of

$$\Phi_\Lambda^\gamma(\vec{z}) := \frac{\Xi_\Lambda(\vec{z})}{\Xi_{\Lambda \setminus \{\gamma\}}(\vec{z})} \,. \tag{5.2a}$$

The second is the *free energy*

$$F_\Lambda(\vec{z}) := -\frac{\log \Xi_\Lambda(\vec{z})}{|\Lambda|} \,. \tag{5.2b}$$

The aim is to find bounds on these quantities independently of Λ, that is in the *thermodynamic limit.* Secondary quantities also of interest are the *pinned connected function* [Far10, (2.24)]

$$\frac{\partial \log \Xi_\Lambda}{\partial z_\gamma}(\vec{z}) \tag{5.2c}$$

and the *rooted connected function* [Far10, (2.25)]

$$z_\gamma \frac{\partial \log \Xi_\Lambda}{\partial z_\gamma}(\vec{z}) \,. \tag{5.2d}$$

Again we want bounds uniformly valid in Λ.

5.1.3 Identities

For $\{\xi_1, \dots, \xi_n\} := \Lambda \Subset \mathcal{P}$ and $\Lambda_i := \{\xi_1, \dots, \xi_i\}$, the *telescoping identity* is

$$\Xi_\Lambda(\vec{z}) = \prod_{i=1}^{n} \Phi_{\Lambda_i}^{\xi_i}(\vec{z}) \,. \tag{5.3a}$$

Even more important is the *fundamental identity* for the partition function:

$$\Xi_\Lambda(\vec{z}) = \Xi_{\Lambda\setminus\{\gamma\}}(\vec{z}) + z_\gamma \, \Xi_{\Lambda\setminus\mathcal{I}(\gamma)}(\vec{z}) \,. \tag{5.3b}$$

In terms of the reduced correlations it looks like

$$\Phi_\Lambda^\gamma(\vec{z}) = 1 + \frac{z_\gamma}{\prod_{i=1}^{m} \Phi_{\Lambda\setminus\{\gamma,\xi_1,\dots,\xi_{i-1}\}}^{\xi_i}(\vec{z})} \,, \tag{5.3c}$$

where $\{\xi_1, \dots, \xi_m\} := \mathcal{I}^\star(\gamma) \cap \Lambda$. The pinned connected function is a product of certain reduced correlations [SS05, (3.8)]:

$$\frac{\partial \log \Xi_\Lambda}{\partial z_\gamma}(\vec{z}) = \frac{\Xi_{\Lambda\setminus\mathcal{I}(\gamma)}(\vec{z})}{\Xi_\Lambda(\vec{z})} = \frac{1}{\Phi_\Lambda^\gamma(\vec{z})} \prod_{i=1}^{m} \frac{1}{\Phi_{\Lambda\setminus\{\gamma,\xi_1,\dots,\xi_{i-1}\}}^{\xi_i}(\vec{z})} \,, \tag{5.3d}$$

where $\{\xi_1, \dots, \xi_m\} := \Lambda \cap \mathcal{I}^\star(\gamma)$.

There is another identity [BFPS11, appendix] expressing the logarithm of the reduced correlations as an integral of the pinned connected function:

$$\log \Phi_\Lambda^\gamma(\vec{z}) = z_\gamma \int_0^1 \frac{\partial \log \Xi_\Lambda}{\partial z_\gamma}(\vec{z}(\alpha)) \, d\alpha \tag{5.3e}$$

with

$$\vec{z}(\alpha) := \begin{cases} z_\xi & \xi \neq \gamma \\ \alpha z_\gamma & \xi = \gamma \,. \end{cases}$$

5.1.4 Cluster expansion

Let I be a finite set. A vector $\vec{\xi} := (\xi_i)_{i\in I} \in \mathcal{P}^I$ has *support*

$$\operatorname{supp} \vec{\xi} := \{\gamma \in \mathcal{P} : \exists\, i \in I : \xi_i = \gamma\} \,. \tag{5.4a}$$

The vector $\vec{\xi}$ *induces* the graph $G(\vec{\xi})$ defined as

$$G(\vec{\xi}) := (I, \{(i,j) \in I^2 : \xi_i \approx \xi_j\}) \,. \tag{5.4b}$$

There are two partitions of I with respect to $G(C_\gamma)$. The first one is the *polymer label partition* $I =: \biguplus_{\gamma\in\operatorname{supp}\vec{\xi}} C_\gamma$. The block C_γ is defined by

$$\forall\, \gamma \in \operatorname{supp} \vec{\xi} : \quad C_\gamma := \{i \in I : \xi_i = \gamma\} \,. \tag{5.4c}$$

The graph $G(C_\gamma)$ is a complete subgraph of $G(\vec{\xi})$. For two distinct polymers $\gamma, \gamma' \in \operatorname{supp} \vec{\xi}$ there are either no edges at all between C_γ and $C_{\gamma'}$, that is

$E(C_\gamma, C_{\gamma'}) = \emptyset$ iff $\gamma \not\approx \gamma'$, or all possible edges are present, that is iff $\gamma \approx \gamma'$.

Therefore the structure of $G(\vec{\xi})$ resembles the one of the polymer subsystem $(\operatorname{supp}\vec{\xi}, \approx)$ – if necessarily zooming out and taken a bird's view. The induced graph $G(\vec{\xi})$ is connected iff the polymer subsystem $(\operatorname{supp}\vec{\xi}, \approx)$ is connected. In this case we call $G(\vec{\xi})$ a *cluster*.

The second partition is the *cluster partition* $I =: \biguplus_{i=1}^{k} C_i$. Each block C_i indexes a connected component, that is a *cluster*, of $G(\vec{\xi})$. In particular the indices in different elements of the partition index mutually compatible polymers, that is for every $C \neq C'$ in the cluster partition of I and each pair of indices $i \in C$ and $j \in C'$ we have $\xi_i \not\approx \xi_j$.

A classic approach to investigate F_Λ is the *Mayer expansion* [MM41] [SS05, section 2.2] or *cluster expansion* [MS10], [Far10, section 2.5]. Define the *Ursell functions* [Urs27](or *semi-invariants* [Dob96a] or *truncated functions* [SS05]) as

$$\mathfrak{u}(\vec{\xi}) := \begin{cases} 1 & |I| = 1 \\ \sum_{H \in \mathcal{C}_{G(\vec{\xi})}} (-1)^{|E(H)|} & |I| \geq 2 \text{ and } G(\vec{\xi}) \text{ connected} \\ 0 & \text{else.} \end{cases} \tag{5.5}$$

The *spanning subgraph complex* $\mathcal{C}_G$ of a graph G is introduced in section 5.4.3. We formally expand the logarithm of the partition function to

$$\log \Xi_\Lambda(\vec{z}) \stackrel{F}{=} \sum_{n \geq 1} \frac{1}{n!} \sum_{\vec{\xi} \in \Lambda^n} \mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} z_{\xi_i} \,. \tag{5.6a}$$

For completeness we show the expansion in section 5.8.3. Other expressions' expansions follow directly:

$$\frac{\partial \log \Xi_\Lambda}{\partial z_\gamma}(\vec{z}) \stackrel{F}{=} \sum_{n \geq 0} \frac{1}{n!} \sum_{\vec{\xi} \in \{\gamma\} \times \Lambda^n} \mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} z_{\xi_i} \tag{5.6b}$$

$$z_\gamma \frac{\partial \log \Xi_\Lambda}{\partial z_\gamma}(\vec{z}) \stackrel{F}{=} \sum_{n \geq 0} \frac{1}{n!} \sum_{\vec{\xi} \in \{\gamma\} \times \Lambda^n} \mathfrak{u}(\vec{\xi}) \prod_{i=0}^{n} z_{\xi_i} \tag{5.6c}$$

$$\log \Phi_\Lambda^\gamma(\vec{z}) \stackrel{F}{=} \sum_{n \geq 1} \frac{1}{n!} \sum_{\vec{\xi} \in \Lambda^n} \mathfrak{u}(\vec{\xi}) [\gamma \in \operatorname{supp}(\vec{\xi})] \prod_{i=1}^{n} z_{\xi_i} \,. \tag{5.6d}$$

To show the convergence of the above series one bounds either (5.6b), (5.6c) or (5.6d). Section 5.1.3 then asserts that all series in (5.6) are finite. A classic strategy to get uniform bounds in Λ is to bound the limit series with Λ replaced by $\mathcal{P}$.

5.1.5 The worst case

The Ursell functions (5.5) have the *alternating sign property*:

$$\forall \vec{\xi} \in \mathcal{P}^{n+1} : \quad (-1)^n \mathfrak{u}(\vec{\xi}) \geq 0 \,. \tag{5.7}$$

This result follows from Penrose's identity in theorem 134. For the general case of a repulsive potential see [SS05, proposition 2.8].

We deduce that worst case is for negative real fugacities (shown in section 5.8.4)

$$\mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} (-|z_{\xi_i}|) \leq \operatorname{Re}\left(\mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} z_{\xi_i}\right) \tag{5.8a}$$

$$\log \Xi_\Lambda(-|\vec{z}|) \leq \operatorname{Re}\left(\log \Xi_\Lambda(\vec{z})\right) \tag{5.8b}$$

$$\Xi_\Lambda(-|\vec{z}|) \leq |\Xi_\Lambda(\vec{z})| \tag{5.8c}$$

and [FP07, (2.10) & (2.11)]

$$\frac{\partial \log \Xi_\Lambda}{\partial z_\gamma}(-|\vec{z}|) \geq \left|\frac{\partial \log \Xi_\Lambda}{\partial z_\gamma}(\vec{z})\right| \tag{5.8d}$$

$$-\log \Phi_\Lambda^\gamma(-|\vec{z}|) \geq |\log \Phi_\Lambda^\gamma(\vec{z})| \tag{5.8e}$$

$$\Phi_\Lambda^\gamma(-|\vec{z}|) \leq |\Phi_\Lambda^\gamma(\vec{z})|\,. \tag{5.8f}$$

Define the *parameter disc* as

$$\mathcal{R}_\mathcal{P} := \left\{\vec{\rho} \in [0,\infty[^\mathcal{P}: \quad \forall\, \Lambda \Subset \mathcal{P}: \quad \Xi_\Lambda(-\vec{\rho}) > 0\right\}. \tag{5.9}$$

5.2 Reaching the aim

From section 5.1.3 we deduce that all quantities of interest (5.2) are expressable in terms of reduced correlations. Thus we look for a lower bound uniform in Λ on $\Phi_\Lambda^\gamma(-\vec{\rho})$. We recall some well-known properties of $\Phi_\Lambda^\gamma(-\vec{\rho})$ and $\mathcal{R}_\mathcal{P}$:

Lemma 117 ([Dob96b],[SS05, section 2.4])**.** *Let $\vec{\rho} \in \mathcal{R}_\mathcal{P}$ and $\gamma \in \Lambda \Subset \mathcal{P}$. Then $\Phi_\Lambda^\gamma(-\vec{\rho})$ decreases monotonously if either Λ or $\vec{\rho}$ increase monotonously. If we introduce more incompatibilities in $\mathcal{P}$ then both $\Phi_\Lambda^\gamma(-\vec{\rho})$ and $\mathcal{R}_\mathcal{P}$ decrease. Finally $\mathcal{R}_\mathcal{P}$ is a* log-convex down-set *and subset of $[0,1]^\mathcal{P}$.*

For completeness we give a proof in section 5.8.5. We can recharacterize $\mathcal{R}_\mathcal{P}$ as

$$\mathcal{R}_\mathcal{P} = \{\vec{\rho} : \forall\, \gamma \in \Lambda \Subset \mathcal{P}: \quad \Phi_\Lambda^\gamma(-\vec{\rho}) \in]0,1]\}\,. \tag{5.10}$$

The upper bound follows from the fundamental identity (5.3c), while the lower bound is just the definition of $\mathcal{R}_\mathcal{P}$. For $\vec{\rho} \in \mathcal{R}_\mathcal{P}$ introduce the well-defined limit of the reduced correlations

$$\Phi_\mathcal{P}^\gamma(-\vec{\rho}) := \inf\{\Phi_\Lambda^\gamma(-\vec{\rho}) : \Lambda \Subset \mathcal{P}\} = \lim_{\Lambda \nearrow \mathcal{P}} \Phi_\Lambda^\gamma(-\vec{\rho}) \in [0,1]\,. \tag{5.11}$$

Therefore, if we have a non-zero lower bound on $\Phi_\Lambda^\gamma(-\vec{\rho})$ for every γ, then we know that $\vec{\rho} \in \mathcal{R}_\mathcal{P}$. In other words

$$\{\vec{\rho} \in [0,1]^\mathcal{P}: \quad \forall\, \gamma \in \mathcal{P} : \Phi_\mathcal{P}^\gamma(-\vec{\rho}) > 0\} \subseteq \mathcal{R}_\mathcal{P}\,.$$

We show in section 5.2.3 that the inclusion is strict.

5.2.1 The classic approach

The standard condition is based on induction on the polymer level. It has been independently discovered several times, most notably by Dobrushin [Dob96b] in the statistical mechanics context and Erdös & Lovász [EL75] under the name of *Lovász Local Lemma* in the context of the probabilistic method. Due to the robustness and simplicity of its proof, the resulting condition is applied in much more general settings [MS00, SS05].

Theorem 118 ([Dob96b], [EL75]). *For $\vec{0} < \vec{\mu} < \vec{\infty}$ let*

$$\forall \gamma \in \mathcal{P}: \qquad \varphi_\gamma^{Dob}(\vec{\mu}) := \prod_{\xi \in \mathcal{I}(\gamma)} (1 + \mu_\xi) \,. \tag{5.12a}$$

If there exists $\vec{\rho} \geq \vec{0}$, such that

$$\vec{\rho}\, \varphi^{Dob}(\vec{\mu}) \leq \vec{\mu} \,, \tag{5.12b}$$

then $\vec{\rho} \in \mathcal{R}_\mathcal{P}$ and we have

$$\forall \gamma \in \mathcal{P}: \qquad \Phi_\mathcal{P}^\gamma(-\vec{\rho}) \geq \frac{1}{1+\mu_\gamma} \,. \tag{5.12c}$$

If $\mathcal{P}$ has uniformly bounded incompatibilities of degree D, then the condition reduces in the homogeneous case to

$$\rho \leq \frac{D^D}{(D+1)^{(D+1)}} \,, \tag{5.13a}$$

which is the maximum of $\frac{\mu}{(1+\mu)^{D+1}}$ at $\mu = 1/D$. Thus

$$\forall \gamma \in \mathcal{P}: \qquad \Phi_\mathcal{P}^\gamma(-\rho) \geq \frac{D}{D+1} \,. \tag{5.13b}$$

Following a more traditional approach Fernández & Procacci [FP07] investigated, for $\gamma \in \mathcal{P}$, the *pinned series*

$$\Psi_\gamma(\vec{\rho}) := \sum_{n \geq 0} \frac{1}{n!} \sum_{\vec{\xi} \in \mathcal{P}^n} |\mathfrak{u}(\gamma, \xi_1, \dots, \xi_n)| \prod_{i=1}^n \rho_{\xi_i} \,. \tag{5.14a}$$

It is the limit series of the cluster expansion of the derivative of the logarithm of the partition function (5.6b) for negative real fugacities. If we apply the identity (5.3d), then we get, for $\vec{\rho} \in \mathcal{R}_\mathcal{P}$ and $\gamma \in \mathcal{P}$, the relation

$$\Psi_\gamma(\vec{\rho}) \leq \frac{1}{\Phi_\mathcal{P}^\gamma(-\vec{\rho})} \prod_{\xi \in \mathcal{I}^\star(\gamma)} \frac{1}{\Phi_\mathcal{P}^{\xi_i}(-\vec{\rho})} \,, \tag{5.14b}$$

Thus the finiteness of $\Psi(\vec{\rho})$ is equivalent to the positivity of $\Phi_\mathcal{P}$. The converse relation follows from (5.3e):

$$\forall \gamma \in \mathcal{P}: \qquad \Phi_\mathcal{P}^\gamma(-\vec{\rho}) \geq \exp\left(-\rho_\gamma \Psi_\gamma(\vec{\rho})\right) \,.$$

Fernández & Procacci's best condition uses tree-operator techniques and is

Theorem 119 ([FP07, proposition 7])**.** *For* $\vec{0} < \vec{\mu} < \vec{\infty}$ *let*

$$\forall\, \gamma \in \mathcal{P}: \qquad \varphi^{FP}_\gamma(\vec{\mu}) := \Xi_{\mathcal{I}(\gamma)}(\vec{\mu})\,, \tag{5.15a}$$

If there exists $\vec{\rho} \geq \vec{0}$*, such that*

$$\vec{\rho}\,\varphi^{FP}(\vec{\mu}) \leq \vec{\mu}\,. \tag{5.15b}$$

then we have $\vec{\rho} \in \mathcal{R}_\mathcal{P}$ *and, for every* $\gamma \in \mathcal{P}$*,*

$$\Psi_\gamma(\vec{\rho}) \leq \frac{\varphi^{FP}_\gamma(\vec{\mu}) - \mu_\gamma}{1-\rho_\gamma} \tag{5.15c}$$

$$\Phi^\gamma_\mathcal{P}(-\vec{\rho}) \geq (1-\rho_\gamma)^{\varphi^{FP}_\gamma(\vec{\mu})-\mu_\gamma}\,. \tag{5.15d}$$

Their proof is discussed in section 5.5. Within their framework they reproduce Dobrushin's result. Comparing (5.15b) and (5.12b) one sees that $\varphi^{\text{Dob}}(\vec{\mu}) \leq \varphi^{\text{FP}}(\vec{\mu})$, whence Fernández & Procacci's result is better. Equality holds for locally tree-like graphs, where $\Xi_{\mathcal{I}(\gamma)}$ factorizes into a product.

5.2.2 The escaping approach

In a number of settings one only wants to show that $\Xi_\Lambda(-\rho) > 0$. One can order every $\Lambda \Subset \mathcal{P}$ such that every reduced correlation in (5.3a) has an unused incompatible polymer. Using the notation from (5.3a) this means, that we can choose the ξ_i such that $\mathcal{I}^\star(\xi) \cap \Lambda_i \neq \mathcal{I}^\star(\xi_i)$, for every $i \in [n]$. Therefore one does not need to control explicitly all reduced correlations. We can trace this idea back to Shearer's calculation on regular rooted tres [She85, theorem 2]. Subsequently it was employed by Liggett, Schonmann & Stacey [LSS97] and Scott & Sokal as "good" pairs [SS05, page 62, paragraph 3].

Definition 120. Call a pair (Λ,γ) *escaping* iff $\gamma \in \Lambda \Subset \mathcal{P}$ and $\mathcal{I}^\star(\gamma) \setminus \Lambda \neq \emptyset$. Call each polymer $\varepsilon \in \mathcal{I}^\star(\gamma) \setminus \Lambda$ an *escape* of (Λ,γ). We also introduce the *escaping pair space*

$$\mathcal{L} := \{(\gamma,\varepsilon) \in \mathcal{P}^2 : \varepsilon \in \mathcal{I}^\star(\gamma)\}\,. \tag{5.16}$$

The above definition is sufficient, because:

Lemma 121. *If* $\Lambda \Subset \mathcal{P}$*, then* Ξ_Λ *is a product of escaping reduced correlations.*

The proof of the lemma is given in section 5.3. The above condition, written in our notation, is

Theorem 122 ([LSS97, theorem 1.3],[SS05, corollary 5.7])**.** *If* $\mathcal{P}$ *has uniformly bounded incompatibilities of degree* D *and if*

$$\rho \leq \frac{(D-1)^{(D-1)}}{D^D}\,, \tag{5.17a}$$

then $\rho\vec{1} \in \mathcal{R}_\mathcal{P}$ *and we have*

$$\forall\,(\gamma,\varepsilon) \in \mathcal{L}: \qquad \Phi^\gamma_{\mathcal{P}\setminus\{\varepsilon\}}(-\rho) \geq \frac{D-1}{D}\,. \tag{5.17b}$$

The bound (5.17a) is optimal for D-regular infinite trees [She85, theorem 2]. We see that $\rho\vec{1} \in \mathcal{R}_\mathcal{P}$ by (5.9). Our first improvement is an inhomogeneous version of theorem 122:

Proposition 123. *For $\vec{0} < \vec{\mu} < \vec{\infty}$ let*

$$\forall\, \gamma \in \mathcal{P} : \qquad \varphi^{esc}_\gamma(\vec{\mu}) := (1+\mu_\gamma) \max_{\varepsilon \in \mathcal{I}^\star(\gamma)} \left\{ \prod_{\xi' \in \mathcal{I}^\star(\gamma)\setminus\{\varepsilon\}} (1+\mu_{\xi'}) \right\}. \tag{5.18a}$$

If there exists $\vec{\rho} \geq \vec{0}$, such that

$$\vec{\rho}\,\varphi^{esc}(\vec{\mu}) \leq \vec{\mu}\,, \tag{5.18b}$$

then $\vec{\rho} \in \mathcal{R}_\mathcal{P}$ and we have

$$\forall\, (\gamma,\varepsilon) \in \mathcal{L} : \qquad \Phi^\gamma_{\mathcal{P}\setminus\{\varepsilon\}}(-\vec{\rho}) \geq \frac{1}{1+\mu_\gamma}\,. \tag{5.18c}$$

The proof is in section 5.3. Proposition 123 is a straightforward combination of the inductive approach pioneered in theorem 118 and the escaping pairs idea employed in theorem 122. Comparing (5.13a) and (5.18b) one sees that $\varphi^{\mathrm{esc}}(\vec{\mu}) \leq \varphi^{\mathrm{Dob}}(\vec{\mu})$, whence the escaping condition improves on Dobrushin's condition in theorem 118. I conjecture that this is the limit of the inductive approach on the polymer level, for two reasons. First, this condition is optimal on all trees for inhomogeneous parameters – this is a consequence of tree-operator theory explained in section 5.6.6. Second, the fundamental identity for the reduced correlations (5.3c) factorizes only over $\mathcal{I}^\star(\gamma)$, thus is unable to catch incompatibilities further away from γ.

Our next step is to see how this result implies bounds for all reduced correlations, by relaxing the parameter a bit.

Proposition 124. *If $\vec{0} \leq \vec{\rho} \leq \vec{\mu} \in \mathcal{R}_\mathcal{P}$, then*

$$\forall\, \gamma \in \mathcal{P} : \qquad \Phi^\gamma_\mathcal{P}(-\vec{\rho}) \geq \begin{cases} \frac{\mu_\gamma - \rho_\gamma}{\mu_\gamma} & \text{if } \mu_\gamma > 0 \\ 1 & \text{if } \mu_\gamma = 0\,. \end{cases} \tag{5.19}$$

Its proof is given in section 5.3. Proposition 124 implies that, up to a small error, uniform control of all reduced correlations is already established by uniform control over all escaping reduced correlations. If we are satisfied with the generic bounds (5.19), then our central question becomes:

Question 125. What are the properties of $\mathcal{R}_\mathcal{P}$? What are sufficient or necessary conditions for $\vec{\rho}$ to be in $\mathcal{R}_\mathcal{P}$?

Let the *interior* of $\mathcal{R}_\mathcal{P}$ with respect to the *box-topology* be

$$\operatorname{Int} \mathcal{R}_\mathcal{P} := \{\vec{\rho} \in \mathcal{R}_\mathcal{P} : \quad \exists\, \vec{x} > \vec{0} : \quad \vec{\rho} + \vec{x} \in \mathcal{R}_\mathcal{P}\}\,, \tag{5.20}$$

Then proposition 124 implies that

$$\vec{\rho} \in \operatorname{Int} \mathcal{R}_\mathcal{P} \Rightarrow (\forall\, \gamma \in \mathcal{P} : \quad \Phi^\gamma_\mathcal{P}(-\vec{\rho}) > 0)\,.$$

We talk about the converse inclusion in section 5.2.3.

In the homogeneous case (5.17a) is better than Dobrushin's (5.13a) and for triangle-free polymer systems even better than Fernández & Procacci's (5.15b). Also the operator φ^{esc} (5.18a) looks like the result of a yet unknown tree-operator in Fernández & Procacci's framework. This immediately suggests to derive and improve the condition via cluster-expansion and tree-operator techniques. We introduce, for $(\gamma, \varepsilon) \in \mathcal{L}$, the *escaped pinned series*

$$\Psi^{\star}_{(\gamma,\varepsilon)}(\vec{\rho}) := \sum_{n\geq 0} \frac{1}{n!} \sum_{\vec{\xi}\in(\mathcal{P}\setminus\{\varepsilon\})^n} |\mathfrak{u}(\gamma, \xi_1, \ldots, \xi_n)| \prod_{i=1}^{n} \rho_{\xi_i} \,. \tag{5.21}$$

Our improvement combining tree-operators and escaping ideas is

Proposition 126. *For $\vec{0} < \vec{\mu} < \vec{\infty}$ let*

$$\forall\, \gamma \in \mathcal{P}: \qquad \varphi^{cTc}_{\gamma}(\vec{\mu}) := (1 + \mu_\gamma) \max\left\{\Xi_{\mathcal{I}^\star(\gamma)\setminus\{\varepsilon\}}(\vec{\mu}) : \varepsilon \in \mathcal{I}^\star(\gamma)\right\} \tag{5.22a}$$

If $\vec{\rho} \geq \vec{0}$, such that

$$\vec{\rho}\, \varphi^{cTc}(\vec{\mu}) \leq \vec{\mu}\,, \tag{5.22b}$$

then $\vec{\rho} \in \mathcal{R}_{\mathcal{P}}$ and we have, for every $(\gamma, \varepsilon) \in \mathcal{L}$,

$$\Psi^{\star}_{(\gamma,\varepsilon)}(\vec{\rho}) \leq \frac{\varphi^{cTc}_{\gamma}(\vec{\mu}) - \mu_\gamma}{1 - \rho_\gamma} \tag{5.22c}$$

$$\Phi^{\gamma}_{\mathcal{P}\setminus\{\varepsilon\}}(-\vec{\rho}) \geq (1 - \rho_\gamma)^{\varphi^{cTc}_{\gamma}(\vec{\mu}) - \mu_\gamma} \,. \tag{5.22d}$$

Its proof is given in section 5.6. The improvement of (5.22b) over (5.18b) is clear, as $\varphi^{\text{cTc}}(\vec{\mu}) \leq \varphi^{\text{esc}}(\vec{\mu})$. We close this section with a remark on the relation between the escaping and general reduced correlations and pinned series. We have

$$\forall\, \gamma \in \mathcal{P}: \qquad \Psi_\gamma(\vec{\rho}) = \Big(1 + \rho_\gamma \Psi_\gamma(\vec{\rho})\Big) \Xi_{\mathcal{I}^\star(\gamma)}\Big(\vec{\rho}\, \Psi^{\star}_{(.,\gamma)}(\vec{\rho})\Big) \,. \tag{5.23a}$$

We can also improve upon (5.14b) by clarifying the exact relation between escaping and non-escaping terms

$$\forall\, \gamma \in \mathcal{P}: \qquad \Psi_\gamma(\vec{\rho}) = \frac{1}{\Phi^{\gamma}_{\mathcal{P}}(-\vec{\rho})} \prod_{\xi\in\mathcal{I}^\star(\gamma)} \frac{1}{\Phi^{\xi_i}_{\mathcal{P}\setminus\{\gamma\}}(-\vec{\rho})} \tag{5.23b}$$

and

$$\begin{aligned} \forall\, (\gamma, \varepsilon) \in \mathcal{L}: \qquad \Psi^{\star}_{(\gamma,\varepsilon)}(\vec{\rho}) &= \frac{1}{\Phi^{\gamma}_{\mathcal{P}\setminus\{\varepsilon\}}(-\vec{\rho})} \prod_{\xi\in\mathcal{I}^\star(\gamma)} \frac{1}{\Phi^{\xi_i}_{\mathcal{P}\setminus\{\varepsilon\}\setminus\{\gamma\}}(-\vec{\rho})} \\ &\leq \frac{1}{\Phi^{\gamma}_{\mathcal{P}\setminus\{\varepsilon\}}(-\vec{\rho})} \prod_{\xi\in\mathcal{I}^\star(\gamma)} \frac{1}{\Phi^{\xi_i}_{\mathcal{P}\setminus\{\gamma\}}(-\vec{\rho})} \,. \end{aligned} \tag{5.23c}$$

5.2.3 The relation to singularity analysis of the homogeneous free energy

The analyticity of the free energy in the thermodynamical limit is a preoccupation in physics. See the works about *transfer matrices* [Gut87, Tod99, Yb11],[Sim93, II.5/II.6] [Bax89] and the comments in [SS05, section 8]. They investigate the location of the *smallest negative singularity* $-\lambda_c$ of

$$z \mapsto F_{\mathcal{P}}(z) := \lim_{\Lambda \nearrow \mathcal{P}} F_\Lambda(z)\,.$$

This is done for *homogeneous fugacities and polymer systems like regular two-dimensional lattices*, that is $(\mathcal{P}, \approx)$ is $\mathbb{Z}^2$ or the hexagonal lattice modulo the loops. The limit is taken along a sequence of exhausting finite sublattices and along a *Følner sequence*, that is in *van Hove* sense. This means that the quotient of the size of the boundary to the size of the sublattice tends to zero, as $\Lambda \nearrow \mathcal{P}$. What is missing from all accounts of singularity analysis known to me is the relation between $-\lambda_c$ and $\mathcal{R}_{\mathcal{P}}$. This section clarifies this and characterizes the singularity as the boundary point of the homogeneous cross-section of $\mathcal{R}_{\mathcal{P}}$. The exact nature of this relation is

Proposition 127. *The multidimensional open (with respect to the box topology) interval* $[\vec{0}, \lambda_c\vec{1}[$ *is a subset of* $\operatorname{Int}\mathcal{R}_{\mathcal{P}}$ *and* $\forall\, \varepsilon > 0 : (\lambda_c + \varepsilon\vec{1}) \notin \operatorname{Int}\mathcal{R}_{\mathcal{P}}$.

and

Proposition 128. *For every* $\Lambda \Subset \mathcal{P}$ *the free energy* F_Λ *is analytic on the set* $\{\vec{z} \in \mathbb{C}^{\mathcal{P}} : \exists\, \vec{\rho} \in \operatorname{Int}\mathcal{R}_{\mathcal{P}} : |\vec{z}| \leq \vec{\rho}\}$.

I want to stress that this section assumes the existence of this singularity. A discussion about the existence of the analytic function $F_{\mathcal{P}}(z)$, for small z and on $\mathbb{Z}^d$, is in [SS05, section 8.3]. We give an alternate proof for $\mathbb{Z}^2$ (with nearest-neighbour edges and loops) in section 6.4.4.

The proofs of propositions 127 and 128 are in section 5.8.6. If $\lambda_c\vec{1}$ lies in $\operatorname{Int}\mathcal{R}_{\mathcal{P}}$, then we can only add vectors $\vec{\varepsilon} > \vec{0}$ with $\inf\{\varepsilon_\gamma : \gamma \in \mathcal{P}\} = 0$ to it, and still have $\lambda_c\vec{1} + \vec{\varepsilon} \in \mathcal{R}_{\mathcal{P}}$. Currently I am not sure, if $\lambda_c\vec{1} \in \operatorname{Int}\mathcal{R}_{\mathcal{P}}$ (with respect to the box topology) or not. More general is

Question 129. Does the statement of proposition 128 extend to $F_{\mathcal{P}}$? I am not good enough at infinite dimensional complex analysis to know if the following is possible:

$$\frac{\partial F_{\mathcal{P}}}{\partial z_\gamma} = \lim_{\Lambda \nearrow \mathcal{P}} \frac{\partial F_\Lambda}{\partial z_\gamma}$$

and conclude by the boundedness of the first (and also all higher) derivatives. It might only be possible with uniform control, that means $\vec{\rho} + \varepsilon\vec{1} \in \mathcal{R}_{\mathcal{P}}$, for some $\varepsilon > 0$. The estimate on $F_\Lambda(-\vec{\rho})$ would then be the same as in the proof of proposition 128. The question is if this is *sufficient* for the analyticity of $F_{\mathcal{P}}$ at $-\vec{\rho}$. My problems start with my lack of knowledge about any definition of analyticity for a countable number of complex variables.

5.3 Polymer level inductive proofs in the escaping approach

The proofs united in this section only use polymer-level induction, that is the fundamental identity (5.3b) (or (5.3c)).

Proof of lemma 121. Let Λ be finite. Recall that we assume that $(\mathcal{P}, \approx)$ is infinite and connected. We can enumerate $\Lambda =: \{\xi_1, \dots, \xi_n\}$ such that every (Λ_j, ξ_j) as defined in the telescoping argument (5.3a) is escaping. This is done the following way: the connectedness of $\mathcal{P}$ implies that there exists a $\xi_1 \in \Lambda$ with an incompatible polymer outside of Λ, that is $|\mathcal{I}^\star(\xi_1) \cap \Lambda| \leq D_{\xi_1} - 1$. Determine ξ_2 by looking at $\Lambda \setminus \{\xi_1\}$ and so on recursively. Hence Ξ_Λ can be written as a product of escaping reduced correlations. □

Proof of proposition 123. We first show (5.18c), for $\gamma \in \Lambda \Subset \mathcal{P}$ with escape ε, by induction over $|\Lambda|$ and then take the limit $\Lambda \nearrow \mathcal{P} \setminus \{\varepsilon\}$. The induction base $\Lambda = \{\gamma\}$ is a trivial case of the following calculation: the expansion by (5.3c) yields an empty product, which equals 1. Let (Λ, γ) be escaping and set $\{\xi_1, \dots, \xi_m\} := \Lambda \cap \mathcal{I}^\star(\gamma)$. Then $m \leq D_\gamma - 1$ and every $(\Lambda \setminus \{\gamma, \xi_1, \dots, \xi_{i-1}\}, \xi_i)$ on the rhs of (5.3c) is escaping, too. Hence

$$
\begin{aligned}
&\Phi^\gamma_\Lambda(-\vec{\rho}) \\
&= 1 - \frac{\rho_\gamma}{\prod_{i=1}^m \Phi^{\xi_i}_{\Lambda \setminus \{\gamma, \xi_1, \dots, \xi_{i-1}\}}(-\vec{\rho})} && \text{by (5.3c)} \\
&\geq 1 - \frac{\rho_\gamma}{\prod_{i=1}^m (1 - \mu_{\xi_i})^{-1}} && \text{by induction} \\
&\geq 1 - \frac{\mu_\gamma}{1 + \mu_\gamma} \times \frac{\min_{\varepsilon \in \mathcal{I}^\star(\gamma)} \prod_{\xi' \in \mathcal{I}^\star(\gamma) \setminus \{\varepsilon\}} (1 + \mu_{\xi'})^{-1}}{\prod_{i=1}^m (1 - \mu_{\xi_i})^{-1}} && \text{by (5.18b)} \\
&\geq 1 - \frac{\mu_\gamma}{1 + \mu_\gamma} \times 1 && \text{cancelling} \\
&= \frac{1}{1 + \mu_\gamma} \,.
\end{aligned}
$$

□

Remark. The following proof, as well as the one of proposition 163, comparing different parameters, applies what I call for lack of a better word the "coupling method" – inspired by a coupling on Shearer's measure [She85]. Something similar has been attempted in [SS05, section 5.2 & 5.3]. But the analysis has been restricted to finite systems with a fixed fugacity vector and the bounds given are not uniform in Λ. The "goodness" condition they use is equivalent to (5.10).

I have an idea how proposition 124 (and in general every hard-repulsion result) extends to the case of soft-repulsion. That is different from the soft-core version of the LLL [SS05, section 4.2], but by comparing a soft-core model with an appropriate hard-core model. This would allow the transfer the results from [FP07] and here immediately to the soft-core case. It would be interesting to know if this would improve on the approach in [Far08], or at least reproduce this result as a corollary.

Proof of proposition 124. We prove the bound (5.19) for all $\gamma \in \Lambda \Subset \mathcal{P}$ by induction over the cardinality of Λ and then take the limit $\Lambda \nearrow \mathcal{P}$. If $\mu_\gamma = 0$, then $0 \leq \rho_\gamma \leq \mu_\gamma = 0$ and $\Phi^\gamma_\Lambda(-\vec{\rho}) = 1 - \rho_\gamma = 1$, independent of the cardinality of Λ. We therefore assume that $\vec{\rho} > \vec{0}$ for the rest of this proof. The induction base with $\Lambda = \{\gamma\}$ is given by

$$\Phi^\gamma_{\{\gamma\}}(-\vec{\rho}) = 1 - \rho_\gamma \geq 1 - \frac{\rho_\gamma}{\mu_\gamma} = \frac{\mu_\gamma - \rho_\gamma}{\mu_\gamma}\,.$$

For the induction step let $\{\xi_1, \dots, \xi_m\} := \mathcal{I}^\star(\gamma) \cap \Lambda$. Then

$$\begin{aligned}
&\Phi^\gamma_\Lambda(-\vec{\mu}) \\
&= 1 - \frac{\mu_\gamma}{\prod_{i=1}^m \Phi^{\xi_i}_{\Lambda \setminus \{\gamma, \xi_1, \dots, \xi_{i-1}\}}(-\vec{\mu})} && \text{by (5.3c)} \\
&= 1 - \frac{\mu_\gamma}{\rho_\gamma} \frac{\rho_\gamma}{\prod_{i=1}^m \Phi^{\xi_i}_{\Lambda \setminus \{\gamma, \xi_1, \dots, \xi_{i-1}\}}(-\vec{\mu})} && \text{we assumed } \rho_\gamma \neq 0 \\
&\geq 1 - \frac{\mu_\gamma}{\rho_\gamma} \frac{\rho_\gamma}{\prod_{i=1}^m \Phi^{\xi_i}_{\Lambda \setminus \{\gamma, \xi_1, \dots, \xi_{i-1}\}}(-\vec{\rho})} && \text{as } \vec{\rho} \leq \vec{\mu} \text{ and by (5.67) (or lemma 117)} \\
&\geq 1 - \frac{\mu_\gamma}{\rho_\gamma}(1 - \Phi^\gamma_\Lambda(-\vec{\rho})) && \text{by (5.3c)}
\end{aligned}$$

Therefore

$$\Phi^\gamma_\Lambda(-\vec{\rho}) \geq \frac{\mu_\gamma - \rho_\gamma}{\mu_\gamma} + \frac{\rho_\gamma}{\mu_\gamma}\Phi^\gamma_\Lambda(-\vec{\mu}) \geq \frac{\mu_\gamma - \rho_\gamma}{\mu_\gamma}\,.$$

□

5.4 The toolbox

This section contains a series of technical definitions and results. They are the building blocks for the following sections. You may want to skip this section on first reading and refer back to it later, when needed.

5.4.1 Lattices and fixed points

In this section we present an adaption of a well known fixed point theorem on lattices by Tarski [Tar55]. Part of the results are inspired by calculations by Faris [Far10] and Fernández & Procacci [FP07].

For some countable set of labels L, let $X := [0, \infty]^L$ be the *lattice* with *partial order* $\vec{x} \leq \vec{y}$, that is coordinate-wise comparison of the vectors: $\forall\, l \in L : x_l \leq y_l$. The *supremum* and *infimum* of a subset A of X are defined by $\sup A := (\sup a_l : \vec{a} \in A)_{l \in L}$ and $\inf A := (\inf a_l : \vec{a} \in A)_{l \in L}$ respectively. We recall that $\sup \emptyset = \vec{0}$ and $\inf \emptyset = \vec{\infty}$.

We say that a function $\phi : X \to X$ *preserves the order* iff

$$\forall\, \vec{x}, \vec{y} \in X : \qquad \vec{x} \leq \vec{y} \Rightarrow \phi(\vec{x}) \leq \phi(\vec{y})\,. \tag{5.24}$$

A sequence $(\vec{y}^{(n)})_{n \in \mathbb{N}}$ of elements of X is said to be *non-decreasing* and *non-increasing* iff $\forall\, n \in \mathbb{N} : \vec{y}^{(n)} \leq \vec{y}^{(n+1)}$ and $\vec{y}^{(n)} \geq \vec{y}^{(n+1)}$ respectively.

Proposition 130. *Let* $(\vec{y}^{(n)})_{n\in\mathbb{N}}$ *be a non-decreasing sequence of elements of* X*. Then* $\lim_{n\to\infty}\vec{y}^{(n)} = \sup_{n\in\mathbb{N}}\vec{y}^{(n)}$*. Same for a non-increasing sequence and its* $\inf$*.*

Remark. The limit above is taken in the *box-topology*, that is the limit is taken independently in each coordinate. Uniform convergence in the product topology is too strong.

Proposition 131. *Let* $\vec{y}\in X$ *and* $\phi : X \to X$ *be order-preserving. If* $(\vec{y}^{(n)} := \phi^n(\vec{y}))_{n\in\mathbb{N}}$ *is a non-decreasing sequence, then* $\phi(\sup_{n\in\mathbb{N}}\vec{y}^{(n)}) = \sup_{n\in\mathbb{N}}\phi(\vec{y}^{(n)})$*. Same for a non-increasing sequence and its* $\inf$*.*

Note to self: Lower semicontinuity of ϕ is a much stronger property. ♢

Proof. Monotone convergence in each coordinate. □

Let $\phi : X \to X$ be order preserving. The set of *fixed points* of ϕ is

$$F_\phi := \{\vec{y}\in X : \phi(\vec{y}) = \vec{y}\} \tag{5.25a}$$

and the set of *decreasing points* is

$$D_\phi := \{\vec{y}\in X : \phi(\vec{y}) \le \vec{y}\}\,. \tag{5.25b}$$

Proposition 132 (after [Tar55, theorem 1] and [Far10, page 177])**.** *Let* $\phi : X \to X$ *be order preserving. Then*

$$\lim_{n\to\infty}\phi^n(\vec{0}) = \inf D_\phi = \inf F_\phi \in F_\phi \neq \emptyset \tag{5.26a}$$

and

$$F_\phi \cap [0,\infty[^L \neq \emptyset \Leftrightarrow D_\phi \cap [0,\infty[^L \neq \emptyset\,. \tag{5.26b}$$

Proof. Let $\vec{z} := \inf D_\phi$. We show that $\vec{z}\in F_\phi$ and $\vec{z} = \inf F_\phi$. We have

$$\begin{aligned}
&\forall\,\vec{y}\in S : \vec{z}\le\vec{y}\\
\Rightarrow\ &\forall\,\vec{y}\in D_\phi : \vec{z} = \phi(\vec{z}) \le \phi(\vec{y}) \le \vec{y}\\
\Rightarrow\ &\phi(\vec{z}) \le \inf D_\phi = \vec{z}\\
\Rightarrow\ &\vec{z}\in D_\phi\\
\Rightarrow\ &\phi^2(\vec{z}) \le \phi(\vec{z})\\
\Rightarrow\ &\phi(\vec{z})\in D_\phi\\
\Rightarrow\ &\vec{z} = \inf D_\phi \le \phi(\vec{z})\,.
\end{aligned}$$

This implies that $\vec{z} = \phi(\vec{z})$, whence $\vec{z}\in F_\phi$. As $F_\phi \subseteq D_\phi$ we deduce that $\inf D_\phi = \inf F_\phi = \vec{z}$.

We show by induction that $\phi^n(\vec{0})$ is non-decreasing. The induction basis is $\phi^0(\vec{0}) = \vec{0} \le \phi(\vec{0})$. The induction step from n to $n+1$ is given by

$$\phi^n(\vec{0}) \le \phi^{n+1}(\vec{0}) \Rightarrow \phi^{n+1}(\vec{0}) = \phi(\phi^n(\vec{0})) \le \phi(\phi^{n+1}(\vec{0})) = \phi^{n+2}(\vec{0})\,.$$

Hence $(\phi^n(\vec{0}))_{n\in\mathbb{N}_0}$ is non-decreasing. Furthermore one can show by induction, that $\forall\, n \in \mathbb{N}_0 : \phi^n(\vec{0}) \leq \vec{z}$, which implies that $\vec{u} := \lim_{n\to\infty} \phi^n(\vec{0}) = \sup_{n\to\infty} \phi^n(\vec{0}) \leq \vec{z}$. Even more, we get

$$\vec{u} = \sup_{n\in\mathbb{N}_0} \phi^n(\vec{0}) = \phi(\sup_{n\in\mathbb{N}_0} \phi^n(\vec{0})) = \phi(\vec{u})\,.$$

Hence $\vec{u} \in F_\phi$ and $\vec{z} = \inf F_\phi \leq \vec{u}$. We conclude that $\vec{z} = \vec{u}$. □

Proposition 133 (after [FP07, proposition 8]). *Let $\phi : X \to X$ be order-preserving and set $\vec{\rho} := \phi(\vec{0})$. If $\vec{\mu} \in D_\phi$, then $(\phi^n(\vec{\rho}))_{n\in\mathbb{N}_0}$ is non-decreasing with limit $\vec{\rho}^\star$, $(\phi^n(\vec{\mu}))_{n\in\mathbb{N}_0}$ is non-increasing with limit $\vec{\mu}^\star$ and we have*

$$\phi(\vec{\rho}^\star) = \vec{\rho}^\star \leq \vec{\mu}^\star = \phi(\vec{\mu}^\star)\,. \tag{5.27}$$

In particular, $\vec{\rho}^\star \in [0,\infty[^L$ iff $D_\phi \cap [0,\infty[^L \neq \emptyset$.

Remark. Proposition 132 asserts that $\vec{\rho}^\star = \inf F_\phi$, that is it is the smallest fixed point of ϕ. There is no a priori reason for $\vec{\rho}^\star = \vec{\mu}^\star$. Later on ϕ is (in each coordinate) a polynomial of finite degree with non-negative powers and coefficients – maybe this allows to show uniqueness of the fixed point?

Proof. As $\vec{0} \leq \vec{\rho}$ we have $\phi(\vec{0}) \leq \phi(\vec{\rho})$. As $\vec{0} \leq \vec{\mu}$ we have $\vec{\rho} = \phi(\vec{0}) \leq \phi(\vec{\mu}) \leq \vec{\mu}$ and $\phi(\vec{\rho}) \leq \phi(\vec{\mu})$. Thus

$$\vec{\rho} \leq \phi(\vec{\rho}) \leq \phi(\vec{\mu}) \leq \vec{\mu}\,.$$

A straightforward alternating induction shows that

$$\forall\, m,n \in \mathbb{N} : \qquad \vec{\rho} \leq \phi^n(\vec{\rho}) \leq \phi^{n+m}(\vec{\rho}) \leq \phi^{n+m}(\vec{\mu}) \leq \phi^m(\vec{\mu}) \leq \vec{\mu}\,.$$

It follows that the two sequences are non-decreasing and non-increasing respectively. Apply propositions 130, 131 and 132 to conclude. □

5.4.2 Sum-product identities

This sections enumerates some *sum-product identities* on formal power series. For the remainder of this section A is a finite index set. Let $f : A \times \mathbb{N}_0 \to \mathbb{C}$ and $g : A \times \mathcal{M} \to \mathbb{C}$, where $\mathcal{M}$ is the set of finite subsets of $\mathbb{N}_0$. We assume that $g(a, M) = f(a, |M|)$. The well-known *Cauchy product* is (with $A = \{1, 2\}$)

$$\left(\sum_{n=0}^{\infty} f(1,n)\right)\left(\sum_{n=0}^{\infty} f(2,n)\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} f(1,m)f(2,n-m)\,. \tag{5.28a}$$

If A has more than two elements, then

$$\prod_{a\in A}\left(\sum_{n=0}^{\infty} f(a,n)\right) = \sum_{n=0}^{\infty} \sum_{\substack{(m_a)_{a\in A}\in\mathbb{N}_0^A \\ \sum_{a\in A} m_a = n}} \prod_{a\in A} f(a, m_a)\,. \tag{5.28b}$$

For exponential formal series we have

$$\prod_{a\in A}\left(\sum_{n=0}^{\infty} \frac{g(a,[n])}{n!}\right) = \sum_{n=0} \frac{1}{n!} \sum_{\substack{(M_a)_{a\in A} \\ \biguplus_{a\in A} M_a = [n]}} \prod_{a\in A} g(a, M_a)\,. \tag{5.28c}$$

If we only sum over non-empty ordered sets, then

$$\prod_{a\in A}\left(\sum_{n=0}^{\infty}\frac{g(a,[n]_0)}{n!}\right)=\sum_{n=|A|}\frac{1}{n!}\sum_{\substack{(M_a)_{a\in A}\\ \forall\, a\in A:\ M_a\neq\emptyset\\ \biguplus_{a\in A} M_a=[n]_0}}\prod_{a\in A} g(a,M_a)\,. \tag{5.28d}$$

The previous identity can be rewritten in terms of f as

$$\prod_{a\in A}\left(\sum_{n=1}^{\infty}\frac{f(a,n)}{n!}\right)=\sum_{n=|A|}\frac{1}{n!}\sum_{\substack{(m_a)_{a\in A}\in\mathbb{N}^A\\ \sum_{a\in A} m_a=n}}\prod_{a\in A}\frac{f(a,m_a)}{m_a!}\,. \tag{5.28e}$$

Finally the distributive law over the cartesian product $B:=\prod_{a\in A} B_a$ is

$$\prod_{a\in A}\left(\sum_{m_a\in B_a} f(a,m_a)\right)=\sum_{B\ni(m_a)_{a\in A}}\prod_{a\in A} f(a,m_a)\,. \tag{5.28f}$$

5.4.3 Penrose's identity

Let $G:=(V,E)$ be a finite, simple graph. Then $\mathcal{C}_G$ is the *set of all spanning subgraphs* of G. Likewise $\mathcal{T}_G$ is the *set of all spanning trees* of G. There is a *natural partial order* on $\mathcal{C}_G$ given by

$$H\leq H' \Leftrightarrow E(H)\subseteq E(H')\,.$$

In particular we can identify $H\in\mathcal{C}_G$ with $E(H)$. For $L,U\in\mathcal{C}_G$ with $L\leq U$ the *Boolean interval* from L to U is defined as

$$[L,U]:=\{H\in\mathcal{C}_G : L\leq H\leq U\}\,.$$

A function $S:\mathcal{T}_G\to\mathcal{C}_G$ is a *partition* (or *Boolean*) *scheme* of $\mathcal{C}_G$ [Pen67, before (5)] iff

$$\mathcal{C}_G:=\biguplus_{\mathbb{T}\in\mathcal{T}_G}[\mathbb{T},S(\mathbb{T})]\,. \tag{5.29}$$

Define the *set of singleton trees of* S (short *S-trees*) by

$$\mathcal{T}_S(G):=\{\mathbb{T}\in\mathcal{T}_G : \mathbb{T}=S(\mathbb{T})\}\,. \tag{5.30}$$

Theorem 134 (Penrose [Pen67, equation (5)])**.** *Let S be a partition scheme of $\mathcal{C}_G$, then*

$$\sum_{H\in\mathcal{C}_G}(-1)^{|E(H)|}=(-1)^{|V|-1}|\mathcal{T}_S(G)|\,. \tag{5.31}$$

The number of S-trees is independent of the choice of S.

Proof. Let $(x_e)_{e\in E}$ be a vector of numbers. Then

$$\begin{aligned}\sum_{H\in\mathcal{C}_G}\prod_{e\in E(H)}x_e &= \sum_{\mathbb{T}\in\mathcal{T}_G}\prod_{e\in E(\mathbb{T})}x_e\sum_{F\subseteq E(S(\mathbb{T}))\setminus E(\mathbb{T})}\prod_{f\in F}x_f\\ &=\sum_{\mathbb{T}\in\mathcal{T}_G}\prod_{e\in E(\mathbb{T})}x_e\prod_{f\in E(S(\mathbb{T}))\setminus E(\mathbb{T})}(1+x_f)\,.\end{aligned}$$

Set all $x_e=-1$. This cancels all the contributions from trees with $\mathbb{T}\neq S(\mathbb{T})$, while for every $\mathbb{T}\in\mathcal{T}_S(G)$ the contribution is $(-1)^{|V|-1}$. ☐

5.4.4 Inductive partition schemes

Let $G := (V, E)$ be a simple graph. The aim of this section is to introduce partition schemes of $\mathcal{C}_G$ described by an *exploration algorithm* 135, selecting a spanning tree from a spanning subgraph, and a sufficient compatibility condition (5.33) on a partition of $E \setminus E(\mathbb{T})$, for a given $\mathbb{T} \in \mathcal{T}_G$, describing the maximal spanning subgraph of G reducing to $\mathbb{T}$.

We call *static information* our knowledge of the structure of G. This includes labellings and orderings of elements of G. We also fix a *root* $o \in V$. In the following we describe the generic version of an inductive partition scheme *Gen*.

Given $H \in \mathcal{C}_G$ the exploration algorithm $\mathcal{E}_{Gen}$ 135 selects a spanning tree of $H \in \mathcal{C}_G$ by starting at the root o and grows the tree iteratively: during each iteration it considers all the nodes neighbouring its boundary, selects some of those to form the new boundary and removes all conflicting edges. This generic prescription guarantees the selection of a spanning tree of H (see proposition 137). We call *dynamic information* all information we discover about H during its exploration.

The complement is to take a tree $\mathbb{T} \in \mathcal{T}_G$ and partition $E \setminus E(\mathbb{T})$ into the *admissible edges* $\mathcal{A}_{Gen}(\mathbb{T})$ and the *conflicting edges* $\mathcal{C}_{Gen}(\mathbb{T})$. If this partition is compatible (5.33) with the exploration algorithm, then we can define a partition scheme *Gen* by the preimage of the $\mathcal{E}_{Gen}$ (see proposition 138). The compatibility is given iff all the dynamic information used in $\mathcal{E}_{Gen}$ to select $\mathbb{T}$ from H is a function of $\mathbb{T}$ (and the static information).

A partition scheme is *inductive* iff it can be described in the above way. The term inductive has been chosen, because the natural proofs for the various properties work by induction over the level of the (partially selected) spanning tree. The motivation for the formalization of inductive partition schemes is that the exploration algorithm 135 advances locally through the graph. Hence we hope to deduce properties of the singleton trees of such a scheme from the local structural properties of G.

It is far from clear if every partition scheme is inductive. A likely counterexample is the minimal weight spanning tree partition scheme in section 5.8.1. Its global description and dependencies have been an obstacle in my search for its local description.

In the rest of this section we formalize the above idea, resulting in the generic inductive partition scheme *Gen*. The gaps in *Gen*'s specification are filled in later sections, resulting in different inductive partition schemes.

Algorithm 135 (*Gen* exploration)**.** Let $H \in \mathcal{C}_G$. We construct a sequence $(H_k)_{k\in\mathbb{N}_0}$ of subgraphs of H starting with $H_0 := H$ and a sequence $(T_k)_{k\in\mathbb{N}_0}$ of subsets of V starting with $T_0 := \{o\}$. We think of T_k as the *explored tree* part of H_k. We construct H_{k+1} from H_k by the following steps:

(gb) Let the *unexplored part* be $U_k := V \setminus T_k$. Let the *potential nodes* P_k

be the set of neighbours of T_k in U_k and the *boundary* B_k be the set of neighbours of U_k in T_k. SELECT a subset S_k of P_k, containing at least one vertex from $C \cap P_k$ for every connected component C of $H_k|_{U_k}$. Call these the *selected nodes.* Set $T_{k+1} := T_k \uplus S_k$.

(gi) Let the *ignored nodes* be $I_k := P_k \setminus S_k$. REMOVE all the edges in $E(B_k, I_k) \cap E(H_k)$.

(gp) For each $v \in S_k$ SELECT $(v, w_v) \in E(B_k, S_k) \cap E(H_k)$.

(gu) For each $v \in S_k$ REMOVE all $(v, w_v) \neq (v, w) \in E(B_k, S_k) \cap E(H_k)$.

(gc) REMOVE all of $E(S_k) \cap E(H_k)$.

Remark. Observe that all steps factorize over connected components of $H_k|_{U_k}$. We can schedule the iterations on different connected components in parallel and in arbitrary order. We can even advance further into one connected component and explore it more or fully without influencing the exploration of the other connected components.

Proposition 136. *The following invariants hold* $\forall\, k \in \mathbb{N}_0$ *in algorithm 135:*

$$H_{k+1} \leq H_k \tag{5.32a}$$
$$H_k|_{T_k} \text{ is a tree} \tag{5.32b}$$
$$H_k \in \mathcal{C}_G \tag{5.32c}$$
$$H_{k+1}|_{T_k} = H_k|_{T_k} \tag{5.32d}$$
$$H_k|_{U_k} = H|_{U_k} \tag{5.32e}$$
$$(E(U_k) \uplus E(U_k, B_k)) \cap E(H_k) = (E(U_k) \uplus E(U_k, B_k)) \cap E(H) \tag{5.32f}$$
$$\forall\, l \geq k : \quad v \in S_k \Leftrightarrow d_{H_l}(o, v) = k \tag{5.32g}$$
$$B_{k+1} \subseteq S_k \tag{5.32h}$$

Remark. We can replace every occurrence of $E(H_k)$ by $E(H)$ in algorithm 135. This follows from invariant (5.32f), which asserts that for each edge there is exactly one iteration of the exploration algorithm during which it is either selected or removed.

Proof. Invariant (5.32a) is clear as we obtain H_{k+1} from H_k by removing edges. Invariant (5.32e) follows from (5.32f). Invariant (5.32d) follows from (5.32b) and the subgraph relationship (5.32a).

(5.32h): **(gi)** ensures that all connections in H_{k+1} between B_k and U_{k+1} go over S_k. By induction these are all connections in H_{k+1} between T_k and U_{k+1}.

We prove the other invariants by induction. They all hold trivially for $k = 0$.

(5.32b): The subgraph $H_{k+1}|_{T_{k+1}}$ consists of $H_{k+1}|_{T_k}$, the vertices S_k and the edge sets $E(T_k, S_k) \cap E(H_{k+1})$ and $E(S_k) \cap E(H_{k+1})$. By (5.32d) and (5.32b) $H_{k+1}|_{T_k} = H_k|_{T_k}$ is a tree. **(gc)** ensures that $E(S_k) \cap E(H_{k+1}) = \emptyset$. **(gp)** and **(gu)** ensure that each $v \in S_k$ has a unique neighbour in T_k in $H_{k+1}|_{T_{k+1}}$. Thus $H_{k+1}|_{T_{k+1}}$ is a tree.

(5.32c): The tree $H_{k+1}|_{T_{k+1}}$ (5.32b) is connected. If $v \in U_{k+1}$, then **(gb)** asserts that there exists a connected component C of $H_k|_{U_k}$ such that $v \in C$. **(gb)** also asserts that there exists a $w \in C \cap S_k$ such that: **(gp)** asserts that there exists a $z \in T_k$ with $z \backsim w$ in H_k and from (5.32e) it follows that $H_{k+1}|_{U_{k+1}} = H_k|_{U_{k+1}}$. Thus $v \leftrightarrow w \leftrightarrow z \leftrightarrow o$ and H_{k+1} is connected.

(5.32f): During iteration k **(gi)**, **(gu)** and **(gc)** only remove edges in $E(B_k, P_k)$ or $E(S_k)$. Thus $E(U_{k+1}) \cap E(H_k)$ and by (5.32g) $E(B_{k+1}, U_{k+1}) \cap E(H_k) = E(S_k, U_{k+1}) \cap E(H_k)$ are subsets of $E(H_{k+1})$. The statement (5.32f) follows by induction over k.

(5.32g): If $v \in T_k$, then $d_{H_{k+1}}(o, v) = d_{H_k}(o, v) = k$, as $H_{k+1}|_{T_k} = H_k|_{T_k}$ by (5.32d). It rests to show that

$$v \in S_k \;\Leftrightarrow\; d_{H_{k+1}}(o, v) = k + 1\,.$$

If $d_{H_{k+1}}(o, v) = k + 1$, then $v \notin T_k := \biguplus_{l=0}^{k} B_k$ and there exists a w with $(v, w) \in E(H_{k+1})$ and $d_{H_{k+1}}(o, w) = k$. Thus $w \in B_k$. By **(gi)** and **(gu)** the only neighbours of B_k in U_k in the graph H_{k+1} are those in S_k.

On the other hand if $v \in S_k$, then, as $H_{k+1}|_{T_{k+1}}$ is a tree (5.32b), it has a unique parent $w \in B_k \subseteq S_{k-1}$ (5.32h). As w is v's only neighbour at distance k from the root in H_{k+1} (5.32g) we have $d_{H_{k+1}}(o, v) = d_{H_{k+1}}(o, w) + 1 = k + 1$. □

Proposition 137. *Algorithm 135 describes a function $\mathcal{E}_{Gen} : \mathcal{C}_G \to \mathcal{T}_G$. It is monotone decreasing, that is $\forall\, H \in \mathcal{C}_G : \mathcal{E}_{Gen}(H) \leq H$.*

Proof. Let $H \in \mathcal{C}_G$ and apply algorithm 135 to it. We observe that if $T_k = V$, then $H_k = H_{k+1}$. But as long as $T_k \neq V$ we have $S_k \neq \emptyset$ (as $H_k \in \mathcal{C}_G$ by (5.32a)) and therefore $(T_k)_{k\in\mathbb{N}_0}$ grows strictly monotone to and then stabilizes in V in at most $|V|$ steps. This implies that the monotone decreasing (5.32a) sequence $(H_k)_{k\in\mathbb{N}_0}$ of subgraphs of H stabilizes in $H_{|V|}$. Finally (5.32b) asserts that $H_{|V|} = H_{|V|}|_{T_{|V|}}$ is a tree and (5.32c) that $H_{|V|} \in \mathcal{C}_H$ and thus $H_{|V|} \leq H$. □

Proposition 138. *Let $\mathcal{E}_{Gen}$ be as in proposition 137. For each $\mathbb{T} \in \mathcal{T}_G$, partition $E(\mathbb{T})$ into the admissible edges $\mathcal{A}_{Gen}(\mathbb{T})$ and the conflicting edges $\mathcal{C}_{Gen}(\mathbb{T})$. If, for each $H \in \mathcal{E}_{Gen}^{-1}(\mathbb{T})$, we have*

$$\forall\, e \in \mathcal{A}_{Gen}(\mathbb{T}) \setminus E(H) : \quad \mathcal{E}_{Gen}((V, E(H) \uplus \{e\})) = \mathbb{T} \tag{5.33a}$$

and

$$\forall\, C \subseteq \mathcal{C}_{Gen}(\mathbb{T}) \setminus E(H) : \quad \mathcal{E}_{Gen}((V, E(H) \uplus C)) \neq \mathbb{T}\,, \tag{5.33b}$$

then the map

$$Gen : \quad \mathcal{T}_G \to \mathcal{C}_G \quad \mathbb{T} \mapsto Gen(\mathbb{T}) := (V, E(\mathbb{T}) \uplus \mathcal{A}_{Gen}(\mathbb{T})) \tag{5.34}$$

is a partition scheme of G with $[\mathbb{T}, Gen(\mathbb{T})] = \mathcal{E}_{Gen}^{-1}(\mathbb{T})$.

Remark. It may also seem that such partition schemes are extensible to infinite graphs – the partial exploration of the graph and the preimage of the partial tree seem to be only dependent on the distance from o. Let

$$\mathcal{A}_{Gen}(\mathbb{T})^{(l)} := \{e := (i, j) \in \mathcal{A}_{Gen}(\mathbb{T}) : i \in L_l, j \in L_k, k \leq l\}$$

Then $\mathcal{A}_{Gen}(\mathbb{T}) = \biguplus l \geq 1 \mathcal{A}_{Gen}(\mathbb{T})^{(l)}$. The proof of the compatibility of $\mathcal{A}_{Gen}(\mathbb{T})^{(l)}$ with $\mathcal{E}_{Gen}$ may use information about the path to the deeper (with respect to $\mathbb{T}$) end i of an edge e. Thus, although we select a tree in $\mathcal{E}_{Gen}$, there may be vertices which are not selected in a finite number of iterations. We present a concrete counterexample in a later section.

Proof. Fix $\mathbb{T} \in \mathcal{T}_G$. For $A \subseteq \mathcal{A}_{Ret}(\mathbb{T})$ and $C \subseteq \mathcal{C}_{Ret}(\mathbb{T})$ let $H_{(A,C)} := (V, E(\mathbb{T}) \uplus A \uplus C)$. We show that $H_{(A,C)} \in [\mathbb{T}, Gen\mathbb{T}]$ iff $C = \emptyset$.

We first assume that $C = \emptyset$. We argue by induction over the cardinality of A. For the induction base with $A = \emptyset$ we have $H_{(\emptyset,\emptyset)} = \mathbb{T}$ and $\mathcal{E}_{Gen}(\mathbb{T}) = \mathbb{T}$. For the induction step consider $A := A' \uplus \{e\}$. By the induction hypothesis $\mathcal{E}_{Gen}(H_{(A',\emptyset)}) = \mathbb{T}$. As $e \in \mathcal{A}_{Gen}(\mathbb{T})$ we apply (5.33a) to see that $\mathcal{E}_{Gen}(H_{(A,\emptyset)}) = \mathbb{T}$, too.

For the second part assume that $C \neq \emptyset$. We have already shown that $\mathcal{E}_{Gen}(H_{(A,\emptyset)}) = \mathbb{T}$. Therefore (5.33b) implies that $\mathcal{E}_{\mathbb{T}}(H_{(A,C)}) \neq \mathbb{T}$.. □

5.4.5 Depth k tree operators

The aim of this section is to show a generic result for depth k recursive constructions of weighted, labelled, finite trees and convergence of certain series of those trees. It generalizes [FP07, proposition 8 and parts of proposition 7].

Let $\mathfrak{T}_n^=$ be the *set of rooted, finite trees of depth* n, $\mathfrak{T}_n^\leq = \biguplus_{m\leq n} \mathfrak{T}_m^=$ the set of rooted, finite trees of depth at most n and $\mathfrak{T}_\infty^\leq = \biguplus_{m\in\mathbb{N}_0} \mathfrak{T}_m^=$ the set of rooted, finite trees. Count by $\mathfrak{p}(t)$ all the automorphisms of t which keep its root invariant. Let $\mathcal{L}$ be a countable set of labels. A function $c : \mathcal{L}^{V(t)} \to [0, \infty[$ is *t-invariant* iff it is invariant under all rooted automorphism of t (these are the automorphisms of t fixing the root of t). We denote by $L_i(t)$ the *i-th level of* t and by $W(t)$ the non-root vertices of t. Finally t_k^v is the *rooted subtree* of t with root v and depth k.

Let $\mathcal{T}_n$ be the set of rooted trees with vertices $[n]_0$ and root 0 and $\mathcal{T}_\infty := \biguplus_{n\geq 0} \mathcal{T}_n$. For $\tau \in \mathcal{T}_\infty$ we denote its unlabelled version by $t(\tau) \in \mathfrak{T}_\infty^\leq$.

Proposition 139. *Fix $k \in \mathbb{N}$. Suppose we have a collection of t-invariant functions $(c_t)_{t\in\mathfrak{T}_k^\leq}$. Let $X := [0, \infty]^{\mathcal{L}}$ and $\vec{\rho} \in X$. Consider the operator $T_{\vec{\rho}} : X \to X$ defined $\forall\, l \in \mathcal{L}$ by*

$$\vec{\mu} \mapsto [T_{\vec{\rho}}(\vec{\mu})]_l := \sum_{t\in\mathfrak{T}_k^\leq} \frac{1}{\mathfrak{p}(t)} \sum_{\vec{\lambda}\in\{l\}\times\mathcal{L}^{|W(t)|}} c_t(\vec{\lambda}) \prod_{\substack{v\in L_i(t)\\ 0\leq i\leq k-1}} \rho_{\lambda_v} \prod_{w\in L_k(t)} \mu_{\lambda_v} \,. \tag{5.35a}$$

If there exists $\vec{\mu} \in X$ such that

$$T_{\vec{\rho}}(\vec{\mu}) \leq \vec{\mu} \,, \tag{5.35b}$$

then the family of series, indexed by $\mathcal{L}$,

$$R_l(\vec{\rho}) := \sum_{n\geq 0} \frac{1}{n!} \sum_{\vec{\lambda}\in\{l\}\times\mathcal{L}^n} \sum_{\tau\in\mathcal{T}_n} \prod_{\substack{i\in[n]_0\\ d_\tau(0,i)=0 \bmod k}} c_{t(\tau_k^i)}(\vec{\lambda}_{V(\tau_k^i)}) \prod_{j=0}^n \rho_{\lambda_j} \tag{5.35c}$$

converges uniformly, as

$$R(\vec{\rho}) = \lim_{n\to\infty} T^n_{\vec{\rho}}(\vec{\rho}) = T_{\vec{\rho}}(R(\vec{\rho})) \leq \vec{\mu}\,. \tag{5.35d}$$

Proof. (of proposition 139) First we rewrite $R_l(\vec{\rho})$ into a sum over unlabelled trees:

$$
\begin{aligned}
&\sum_{n\geq 0}\frac{1}{n!}\sum_{\vec{\lambda}\in\{l\}\times\mathcal{L}^n}\sum_{\tau\in\mathcal{T}_n}\prod_{\substack{i\in[n]_0\\ d_\tau(0,i)=0 \bmod k}} c_{t(\tau_k^i)}(\vec{\lambda}_{V(\tau_k^i)})\prod_{j=0}^{n}\rho_{\lambda_j}\\
&=\sum_{n\geq 0}\sum_{\tau\in\mathcal{T}_n}\frac{1}{|W(\tau)|!}\sum_{\vec{\lambda}\in\{l\}\times\mathcal{L}^{|W(\tau)|}}\prod_{\substack{i\in[|W(\tau)|]_0\\ d_\tau(0,i)=0 \bmod k}} c_{t(\tau_k^i)}(\vec{\lambda}_{V(\tau_k^i)})\prod_{j=0}^{|W(\tau)|}\rho_{\lambda_j}\\
&=\sum_{\tau\in\mathcal{T}_\infty}\left[\frac{1}{|W(\tau)|!}\prod_{\substack{i\in[|W(\tau)|]_0\\ d_\tau(0,i)=0 \bmod k}}\mathfrak{p}(t(\tau_k^i))\right]\sum_{\vec{\lambda}\in\{l\}\times\mathcal{L}^{W(t(\tau))}}\prod_{m=0}^{\infty}\prod_{v\in L_{mk}(t(\tau))}\frac{c_{t_k^v}(\vec{\lambda}_{V(t_k^v)})}{\mathfrak{p}(t_k^v)}\prod_{w\in V(t(\tau))}\rho_{\lambda_w}\\
&=\sum_{\tau\in\mathcal{T}_\infty}\left[\frac{1}{|W(t(\tau))|!}\prod_{m=0}^{\infty}\prod_{v\in L_{mk}(t(\tau))}\mathfrak{p}(t_k^i)\right]\sum_{\vec{\lambda}\in\{l\}\times\mathcal{L}^{W(t(\tau))}}\prod_{m=0}^{\infty}\prod_{v\in L_{mk}(t(\tau))}\frac{c_{t_k^v}(\vec{\lambda}_{V(t_k^v)})}{\mathfrak{p}(t_k^v)}\prod_{w\in V(t(\tau))}\rho_{\lambda_w}\\
&=\sum_{t\in\mathfrak{T}_\infty^{\lesssim}}\sum_{\vec{\lambda}\in\{l\}\times\mathcal{L}^{W(t)}}\prod_{m=0}^{\infty}\prod_{v\in L_{mk}(t)}\frac{c_{t_k^v}(\vec{\lambda}_{V(t_k^v)})}{\mathfrak{p}(t_k^v)}\prod_{w\in V(t(\tau))}\rho_{\lambda_w}
\end{aligned}
$$

(Proof continued on following landscape page)

(Continuation of proof of proposition 139) By induction we see that $\forall\, n \in \mathbb{N}, l \in \mathcal{L}$:

$$
\begin{aligned}
&[T^n_{\vec\rho}(\vec\mu)]_l \\
&= \rho_l \sum_{t \in \mathfrak{T}^{\leq}_{nk}} \sum_{\vec\lambda \in \{l\} \times \mathcal{L}^{W(t)}} \times \left[\sum_{m=0}^{n-1} \sum_{v \in L_{mk}(t)} \frac{c_{t^v_k}(\vec\lambda_{W(t^v_k)})}{\mathfrak{p}(t^v_k)} \prod_{w \in W(t^v_k)} \rho_{\lambda_w} + \sum_{v \in L_{(n-2)k}(t)} \frac{c_{t^v_k}(\vec\lambda_{W(t^v_k)})}{\mathfrak{p}(t^v_k)} \prod_{i=1}^{k-1} \prod_{w \in L_i(t^v_k)} \rho_{\lambda_w} \prod_{w \in L_k(t^v_k)} \mu_{\lambda_w} \right] \\
&= \sum_{t \in \mathfrak{T}^{\leq}_{nk}} \sum_{\vec\lambda \in \{l\} \times \mathcal{L}^{W(t)}} \sum_{m=0}^{n-1} \sum_{v \in L_{mk}(t)} \frac{c_{t^v_k}(\vec\lambda_{W(t^v_k)})}{\mathfrak{p}(t^v_k)} \prod_{i=0}^{nk-1} \prod_{v \in L_i(t)} \rho_{\lambda_v} \prod_{w \in L_{nk}(t)} \mu_{\lambda_w} \,.
\end{aligned}
$$

This implies that $\forall\, l \in \mathcal{L}$:

$$
[T^n_{\vec\rho}(\vec\rho)]_l = \sum_{t \in \mathfrak{T}^{\leq}_{nk}} \sum_{\vec\lambda \in \{l\} \times \mathcal{L}^{W(t)}} \sum_{m=0}^{n-1} \sum_{v \in L_{mk}(t)} \frac{c_{t^v_k}(\vec\lambda_{W(t^v_k)})}{\mathfrak{p}(t^v_k)} \prod_{v \in V(t)} \rho_{\lambda_v} \,.
$$

Taking the limit we see that

$$
R(\vec\rho) = \lim_{n \to \infty} [T^n_{\vec\rho}(\vec\rho)] \,.
$$

The operator $T_{\vec\rho}$ has only non-negative coefficients, hence is order preserving. Apply proposition 133 together with condition (5.35b) to see that $R(\vec\rho)$ is the least fixed point of $T_{\vec\rho}$ and that (5.35d) holds. □

Remark. Denote by $c_n : X \to \{0,1\}$ the index function of the depth 1 tree consisting of the root and n children. Hence c_0 is the index function of the tree consisting just of the root, c_1 the one of the tree consisting of the root and one child, ...

Proposition 140. *In the setting of proposition 139: if $\forall\, l \in L$: $c_0(l) \geq 1$ and $c_1(l,l) \geq 1$, then $T_{\vec{\rho}} \geq \vec{\rho}(\vec{1} + id_X)$.*

Proof. For every $l \in L$ we have $[T_{\vec{\rho}}(\vec{\mu})]_l \geq c_0(l)\rho_l + c_1(l,l)\mu_l\rho_l \geq \rho_l(1+\mu_l)$ by omitting all trees except the one with just the root, labelled with l, and the other with the root and one child, both labelled with l. □

Proposition 141 (after [BFPS11, appendix])**.** *Suppose that $T_{\vec{\rho}} \geq \vec{\rho}(\vec{1}+\vec{\mu})$ and (5.35b) holds. Decompose $T_{\vec{\rho}} =: \vec{\rho}\,(id_X + S_{\vec{\rho}})$ by splitting off the root of each tree. Likewise decompose $R(\vec{\rho}) =: \vec{\rho}Q(\vec{\rho})$. Then $S_{\vec{\rho}} \geq \vec{0}$ is order-preserving, $\vec{\rho} < \vec{1}$ and we refine (5.35d) to*

$$\forall\, l \in \mathcal{L}: \quad [Q(\vec{\rho})]_l \leq \frac{[S_{\vec{\rho}}(\vec{\mu})]_l}{1-\rho_l}\,. \tag{5.36}$$

Proof. As $R(\vec{\rho})$ and $Q(\vec{\rho})$ increase in $\vec{\rho}$ we can assume without loss of generality that $\vec{\mu} \geq \vec{\rho} > 0$ or fall back on a smaller polymer system omitting the 0 entries of $\vec{\rho}$. We also have

$$\vec{\rho}\,(\vec{1}+\vec{\mu}) \leq T_{\vec{\rho}}(\vec{\mu}) \leq \vec{\mu}\,,$$

whereby

$$\forall\, l \in \mathcal{L}: \quad \rho_l \leq \frac{\mu_l}{1+\mu_l} < 1\,.$$

Thus

$$\vec{\rho}Q(\vec{\rho}) = R(\vec{\rho}) = T_{\vec{\rho}}(R(\vec{\rho})) = \vec{\rho}\,(\mathrm{id}_X + S_{\vec{\rho}})(R(\vec{\rho})) = \vec{\rho}\,R(\vec{\rho}) + S_{\vec{\rho}}(R(\vec{\rho}))\,,$$

whence, ignoring 0 entries of $\vec{\rho}$ and as $S_{\vec{\rho}}$ is order-preserving,

$$Q(\vec{\rho}) = \vec{\rho}\,Q(\vec{\rho}) + S_{\vec{\rho}}(R(\vec{\rho})) \leq \vec{\rho}\,Q(\vec{\rho}) + S_{\vec{\rho}}(\vec{\mu})\,.$$

□

5.4.6 Depth k tree approximation

This section extends [FP07, proposition 7] to level k approximations. For basic notation see section 5.4.5.

Proposition 142. *Fix $k \in \mathbb{N}$ and a polymer system $\mathcal{P}$. Suppose that we have a family of t-invariant functions $(c_t)_{t\in\mathfrak{T}_k^{\leq}}$ such that*

$$\forall\, \vec{\xi} \in \mathcal{P}^{n+1}: \quad |\mathfrak{u}(\xi_0,\dots,\xi_n)| \leq \sum_{\tau\in\mathcal{T}_n} \prod_{\substack{i\in[n]_0\\ d_\tau(0,i)=0 \bmod k}} c_{t(\tau_k^i)}(\vec{\xi}_{V(\tau_k^i)})\,. \tag{5.37a}$$

Let $X := [0,\infty]^{\mathcal{P}}$ and $\vec{\rho} \in X$. Consider the operator $T_{\vec{\rho}} : X \to X$ defined $\forall\, \gamma \in \mathcal{P}$ by

$$\vec{\mu} \mapsto [T_{\vec{\rho}}(\vec{\mu})]_\gamma := \sum_{t\in\mathfrak{T}_k^{\leq}} \frac{1}{\mathfrak{p}(t)} \sum_{\vec{\xi}\in\{\gamma\}\times\mathcal{P}^{|W(t)|}} c_t(\vec{\xi}) \prod_{\substack{v\in L_i(t)\\ 0\leq i\leq k-1}} \rho_{\xi_v} \prod_{w\in L_k(t)} \mu_{\xi_v}\,. \tag{5.37b}$$

If there exists $\vec{\mu} \in X$ *such that*

$$T_{\vec{\rho}}(\vec{\mu}) \leq \vec{\mu}\,, \tag{5.37c}$$

then

$$\vec{\rho}\,\Psi(\vec{\rho}) \leq \vec{\mu}\,. \tag{5.37d}$$

Even better, the condition of proposition 140 applies, and thus

$$\forall\, \gamma \in \mathcal{P} : \quad [\Psi(\vec{\rho})]_l \leq \frac{[T_{\vec{\rho}}(\vec{\mu})]_\gamma - \rho_\gamma \mu_\gamma}{\rho_\gamma(1-\rho_\gamma)}\,. \tag{5.37e}$$

Remark. For $k=1$ the cluster $G(\gamma,\gamma,\gamma)$ can be counted by a two-expansion of a level 1 tree or by $c_2(\gamma,\gamma,\gamma)$. We therefore can not expand (5.37e) further without loss of generality.

Proof. Setting $\mathcal{L} := \mathcal{P}$ in (5.35c) condition (5.37a) implies that

$$\begin{aligned}[\vec{\rho}\,\Psi(\vec{\rho})]_\gamma &= \sum_{n\geq 0} \frac{1}{n!} \sum_{\vec{\xi}\in\{\gamma\}\times\mathcal{P}^n} |\mathfrak{u}(\vec{\xi})| \prod_{i=0}^{n} \rho_{\xi_i} \\ &\leq \sum_{n\geq 0} \frac{1}{n!} \sum_{\vec{\xi}\in\{\gamma\}\times\mathcal{P}^n} \sum_{\tau\in\mathcal{T}_n} \prod_{\substack{i\in[n]_0 \\ d_\tau(0,i)=0 \bmod k}} c_{t(\tau_k^i)}(\vec{\xi}_{V(\tau_k^i)}) \prod_{j=0}^{n} \rho_{\xi_j} = [R(\vec{\rho})]_\gamma\,.\end{aligned}$$

Then (5.37d) follows by applying (5.35d) from proposition 142.

We know that $[T_{\vec{\rho}}(\vec{\mu})]_\gamma$ counts the clusters $G(\gamma)$ and $G(\gamma,\gamma)$ as trees of depth 0 and 1 respectively. Hence $c_0(\gamma)=1$ and $c_1(\gamma,\gamma)=1$. We apply propositions 140 and 141 to get (5.37e).

□

5.5 Review of the approach by Fernández & Procacci

5.5.1 A short synopsis of the tree-operator approach

This section gives a short summary of the main steps of the tree-operator approach. It is intended as coarse big picture of how the different results in sections 5.4 and 5.5 fit together.

- The aim is to majorize the cluster expansion (5.6b) uniformly in Λ.
- This is equivalent to the convergence of the pinned series (5.14a).
- Choose a partition scheme (5.29) and partition the spanning graph complex of a cluster using Penrose's theorem 134.
- We are left with a sum over the singleton trees of the partition scheme on a cluster (5.45).
- Determine some local necessary properties of those singleton trees based on polymer labels (5.42).

- Keep only those local necessary properties, which allows us to decompose each tree as built from trees from a finite set (see proposition 142).
- Reorganize our sums to write the pinned series as the fixpoint of a tree-extension operator based on the above finite sets of trees (proposition 139).
- Control the convergence of this tree-operator by an appropriate condition (weighted sum over the finite set of trees) (5.35b)/(5.45) over an appropriate label space.

5.5.2 Penrose (or greedy) scheme

Let I be a finite and totally ordered set. Let $G := (I, E)$ be a connected graph. We present a formulation of the first partition scheme by Penrose [Pen67, equation (6)] or [FP07, section 4.1] in the inductive style of section 5.4.4. The static information comprises the total order on I, the structure of G and the choice of the root $o \in I$.

Algorithm 143 (*Pen* exploration)**.** Let $H \in \mathcal{C}_G$. For every k let H_k, T_k, U_k, B_k and P_k be as in algorithm 135. The missing specification to go from H_k to H_{k+1} on a connected component C of $H_k|_{U_k}$ is:

(pb) SELECT $C \cap S_k := C \cap P_k$.

(pi) As $C \cap I_k = \emptyset$ REMOVE nothing.

(pp) For $i \in C \cap S_k$ let $j_i := \operatorname{argmin}\{j \in B_k : (i,j) \in E(H_k)\}$. SELECT (i, j_i).

(pu) For $i \in C \cap S_k$ REMOVE all $(i,j) \in E(H_k)$ with $j_i \neq j \in B_k$.

(pc) REMOVE all of $E(C \cap S_k) \cap E(H_k)$.

Remark. We call *Pen* also *greedy* because of the choice of S_k in **(pb)**. Algorithm 143 *flood-fills* H. This means it incorporates a parallel version of *Dijkstra's single-source shortest path algorithm* [Dij59],[KV06, page 145] on a graph with unit edge weights.

Proposition 144. *The function $\mathcal{E}_{Pen}$ described in algorithm 143 is $\mathcal{C}_G \to \mathcal{T}_G$ and monotone decreasing, that is $\forall\, H \in \mathcal{C}_G : \mathcal{E}_{Pen}(H) \leq H$.*

Proof. Follows from proposition 137. □

Algorithm 145 (*Pen* tree edge complement partition)**.** Let $\mathbb{T} \in \mathcal{T}_G$. Let L_k be the k^{th} level of $\mathbb{T}$. We partition $E \setminus E(\mathbb{T})$ into $\mathcal{A}_{Pen}(\mathbb{T}) \uplus \mathcal{C}_{Pen}(\mathbb{T})$. Let $0 \leq k \leq l$, $j \in L_k$, $i \in L_l$ and $e := (i,j) \in E \setminus E(\mathbb{T})$. Then $e \in \mathcal{C}_{Pen}(\mathbb{T})$ iff one of the mutually exclusive conditions (5.38) holds:

$$l \geq k + 2\,, \tag{5.38a}$$

$$l = k + 1 \wedge j < \mathfrak{p}(i)\,. \tag{5.38b}$$

And $e \in \mathcal{A}_{Pen}(\mathbb{T})$ iff one of the mutually exclusive conditions (5.39) holds:

$$k = l\,, \tag{5.39a}$$

$$l = k + 1 \wedge j > \mathfrak{p}(i)\,. \tag{5.39b}$$

Proposition 146. *The map*

$$Pen: \quad \mathcal{T}_G \to \mathcal{C}_G \quad \mathbb{T} \mapsto Pen(\mathbb{T}) := (I, E(\mathbb{T}) \uplus \mathcal{A}_{Pen}(\mathbb{T})) \tag{5.40}$$

is a partition scheme of G *with* $[\mathbb{T}, Pen(\mathbb{T})] = \mathcal{E}_{Pen}^{-1}(\mathbb{T})$.

Proof. If we admit that the partition in algorithm 145 satisfies (5.33), then proposition 146 is a direct consequence of proposition 138. Thus we show (5.33) for $\mathcal{E}_{Pen}$ and $\mathcal{A}_{Pen}(\mathbb{T})$.

Fix $\mathbb{T} \in \mathcal{T}_G$ and $H \in \mathcal{E}_{Ret}^{-1}(\mathbb{T})$. Let $0 \le k \le l$, $j \in L_k$, $i \in L_l$ and $e := (i,j) \in E \setminus E(\mathbb{T})$. Let $(H_n)_{n\in\mathbb{N}_0}$ and $(T_n)_{n\in\mathbb{N}_0}$ be the sequences associated with H from algorithm 151. For $\emptyset \ne F \subseteq E \setminus E(H)$ set $\tilde{H} := (I, E(H) \uplus F)$ and let $(\tilde{H}_n)_{n\in\mathbb{N}_0}$ and $(\tilde{T}_n)_{n\in\mathbb{N}_0}$ be its associated sequences from algorithm 151.

For $e \in F$ let $m(e)$ be the level of $i \curlywedge j$. We claim that for each $N \in \mathbb{N}$:

$$(\forall\, e \in F: m(e) \ge N) \Rightarrow (H_n|_{T_n})_{n=0}^N = (\tilde{H}_n|_{\tilde{T}_n})_{n=0}^N\,. \tag{5.41}$$

We show (5.41) by induction over n. By definition $H_0|_{T_0} = \tilde{H}_0|_{\tilde{T}_0}$. For the induction step from $n < N$ to $n+1$ we show that algorithm 143 is not influenced by the presence of such an $e \in F$. The presence of e does not change $\tilde{P}_n$ compared to P_n in **(pb)**.

Observe that if $e \in \mathcal{A}_{Ret}(\mathbb{T})$, then by (5.39) $m(e) = k$. To show (5.33a) we assume that $F \subseteq \mathcal{A}_{Ret}(\mathbb{T})$. If e is of type (5.39a), then it is removed during iteration $(k-1)$ by **(pc)**. If e is of type (5.39b), then it is removed by **(pu)** during iteration k.

To show (5.33b) we assume that $F' := F \cap \mathcal{C}_{Pen}(\mathbb{T}) \ne \emptyset$. Choose $e \in F$ with $m(e) = N := \min\{m(f) : f \in F'\}$. We demonstrate that the presence of e causes $(H_n|_{T_n})_{n\in\mathbb{N}_0}$ to diverge from $(\tilde{H}_n|_{\tilde{T}_n})_{n\in\mathbb{N}_0}$ exactly at level $N+1$, that is (5.41) holds and $H_{N+1}|_{T_{N+1}} \ne \tilde{H}_{N+1}|_{\tilde{T}_{N+1}}$. We go through all the cases of (5.38):

Case e of type (5.38a): Here $N = k$. The presence of e lets **(pb)** SELECT $i \in \tilde{P}_k$.

Case e of type (5.38b): Here $N = k$. The presence of e lets **(pp)** SELECT i as the parent of i in $\tilde{T}_{N+1}$ instead of $\mathfrak{p}(i)$. □

We specialize to the case of $G := G(\vec{\xi})$ being the cluster induced by a vector $\vec{\xi} \in \mathcal{P}^I$. This increases the static information about G – its structure is a function of $\vec{\xi}$.

Proposition 147 (Properties of $\mathcal{T}_{Pen}(G(\vec{\xi}))$)**.** *Let* $\mathbb{T} \in \mathcal{T}_{Pen}(G(\vec{\xi}))$ *and let* C_i *be the set of children of* i *in* $\mathbb{T}$. *Then*

$$\forall\, k \in \mathbb{N}_0: \quad \operatorname{supp} \vec{\xi}_{L_k} \text{ is an independent subset of } \mathcal{P} \tag{5.42a}$$

$$\operatorname{supp} \vec{\xi}_{C_i} \text{ is an independent subset of } \mathcal{I}(\xi_i) \tag{5.42b}$$

$$|C_i| = |\operatorname{supp} \vec{\xi}_{C_i}|\,. \tag{5.42c}$$

Proof. Fix k. Suppose we have $i, j \in L_k$ with $\xi_i \approx \xi_j$, then $e := (i,j) \in E$ as $G = G(\vec{\xi})$ is the cluster of $\vec{\xi}$. Hence by (5.39a) $e \in E(Pen\mathbb{T})$ and thus $Pen(\mathbb{T}) \neq \mathbb{T}$. This shows (5.42a), which implies (5.42b), which in turn implies (5.42c). □

5.5.3 Depth one tree approximation

We use the notation for trees from section 5.4.6. The summary of Fernández & Procacci's tree approximation machinery is:

Theorem 148 ([FP07, proposition 7]). *Let $(c_n)_{n\in\mathbb{N}_0}$ be a sequence of functions, where each $c_n : \gamma^{n+1} \to \{0,1\}$ is invariant under permutations of its last n arguments. Suppose that*

$$\forall \vec{\xi} \in \mathcal{P}^{n+1} : \quad |\mathfrak{u}(\xi_0, \dots, \xi_n)| \leq \sum_{\tau\in\mathcal{T}_n} \prod_{i\in[n]_0} c_{s_i}(\xi_i, \xi_{i_1}, \dots, \xi_{i_{s_i}})\,. \tag{5.43a}$$

Let $X := [0,\infty]^{\mathcal{P}}$ and $\vec{\rho} \in X$. Consider the operator $T_{\vec{\rho}} : X \to X$ defined by $T_{\vec{\rho}} := \vec{\rho}\,\varphi^{gen}$, where $\varphi^{gen} : X \to X$ and

$$\forall \gamma \in \mathcal{P} : \qquad \varphi^{gen}_\gamma(\vec{\mu}) := \sum_{n\geq 0} \frac{1}{n!} \sum_{\vec{\xi}\in\{\gamma\}\times\mathcal{P}^n} c_n(\vec{\xi}) \prod_{i=1}^{n} \mu_{\xi_i}\,. \tag{5.43b}$$

If there exists $\vec{\mu} \in X$ such that

$$T_{\vec{\rho}}(\vec{\mu}) = \vec{\rho}\,\varphi^{gen}(\vec{\mu}) \leq \vec{\mu}\,, \tag{5.43c}$$

then

$$\vec{\rho}\,\Psi(\vec{\rho}) \leq \vec{\mu} \tag{5.43d}$$

and $\vec{\rho} \in \mathcal{R}_{\mathcal{P}}$. Suppose φ^{gen} fulfils also

$$\widehat{\varphi^{gen}} := \varphi^{gen} - id_X \geq 0\,, \tag{5.44a}$$

then we have for every $\gamma \in \mathcal{P}$:

$$\Psi_\gamma(\vec{\rho}) \leq \frac{\widehat{\varphi^{gen}_\gamma}(\vec{\mu})}{1-\rho_\gamma} \tag{5.44b}$$

$$\log \Phi^\gamma_{\mathcal{P}}(-\vec{\rho}) \geq \widehat{\varphi^{gen}_\gamma}(\vec{\mu}) \log(1-\rho_\gamma) \tag{5.44c}$$

and for each $\gamma \in \Lambda \Subset \mathcal{P}$

$$\Phi^\gamma_\Lambda(-\vec{\rho}) \geq (1-\rho_\gamma)^{\widehat{\varphi^{gen}_\gamma}(\vec{\mu})} \tag{5.44d}$$

$$\Xi_\Lambda(-\vec{\rho}) \geq \prod_{\xi\in\Lambda} (1-\rho_\xi)^{\widehat{\varphi^{gen}_\xi}(\vec{\mu})}\,. \tag{5.44e}$$

Proof. Part (5.43) is a corollary to the more general proposition 142.

For part (5.44) the first bound (5.44b) follows from (5.37e). For (5.44c) use identity (5.3e). The bounds (5.44d) and (5.44e) are direct consequences of the prior majorations. □

Remark. Condition (5.44a) is natural: looking at (5.43b) we can see that

$$\varphi_\gamma^{\text{gen}}(\vec{\mu}) \geq c_1(\gamma,\gamma)\mu_\gamma = \mu_\gamma\,,$$

as we have to count the cluster $G(\gamma,\gamma)$. Furthermore condition (5.44a) guarantees the well-definedness of (5.44b):

$$\forall\,\gamma \in \mathcal{P}: \quad \left(\rho_\gamma(1+\mu_\gamma) < \rho_\gamma\varphi_\gamma^{\text{gen}}(\vec{\mu}) \leq \mu_\gamma \;\Rightarrow\; \rho_\gamma \leq \frac{\mu_\gamma}{1+\mu_\gamma} < 1\right)\,,$$

by counting also the cluster $G(\gamma)$.

The best sufficient condition derived by Fernández & Procacci is

$$\forall\,\gamma \in \mathcal{P}: \quad \varphi_\gamma^{\text{FP}}(\vec{\mu}) := \Xi_{\mathcal{I}(\gamma)}(\vec{\mu})\,. \tag{5.45}$$

It results from applying Penrose's identity (5.31) with Penrose's partition scheme *Pen* described in section 5.5.2:

$$\begin{aligned}
&|\mathfrak{u}(\xi_0,\dots,\xi_n)|\\
&= |\mathcal{T}_{Pen}(G(\xi_0,\dots,\xi_n))|\\
&= \sum_{\tau\in\mathcal{T}_n} [\tau \in \mathcal{T}_{Pen}(G(\xi_0,\dots,\xi_n))]\\
&\leq \sum_{\tau\in\mathcal{T}_n} \prod_{i\in[n]_0} c_{s_i}^{\text{FP}}(\xi_i,\xi_{i_1},\dots,\xi_{i_{s_i}})\,.
\end{aligned}$$

The c_n^{FP}s in the last line are a straight encoding of the properties enounced in proposition 147, namely

$$c_n^{\text{FP}}(\xi_0,\dots,\xi_n) := \prod_{i=1}^{n} [\xi_0 \approx \xi_i] \prod_{i\neq j=1}^{n} [\xi_j \not\approx \xi_i]\,.$$

The condition (5.43c) admits a bigger $\vec{\rho}$ iff the majoration (5.43a) is tighter. This framework incorporates previous conditions [FP07, (3.7)], for example the condition (5.12b) of Dobrushin's approach can be written as

$$\forall\,\gamma \in \mathcal{P}: \quad \varphi_\gamma^{\text{Dob}}(\vec{\mu}) := \prod_{\xi\in\mathcal{I}(\gamma)} (1+\mu_\xi)\,. \tag{5.46}$$

This comes from relaxing the condition in the c_n^{FP}s to

$$c_n^{\text{Dob}}(\xi_0,\dots,\xi_n) := \prod_{i=1}^{n} [\xi_0 \approx \xi_i] \prod_{i\neq j=1}^{n} [\xi_j \neq \xi_i]\,.$$

5.5.4 A modified Penrose scheme for higher depth approximations

The aim of this section is to show how to encode the uncle conditions implied by **(pu)** in the family $\mathfrak{T}_k^{=}$ instead of $\mathcal{T}_n$, thus ignoring the ordering of the vertices implied by their bijective labelling by $[n]_0$. Uncle conditions only play a role in approximations of depth 2 or higher.

One approach is to modify Pen on a cluster $G(\vec{\xi})$ to keep more information about the polymers involved. Fix a collection $\{<_\gamma : \gamma \in \mathcal{P}\}$ of total orders $<_\gamma$ on $\mathcal{I}(\gamma)$.

Algorithm 149 ($PenM$ exploration)**.** We only list the differences to algorithm 143.

Let C be a connected component C of $H_k|_{U_k}$. Then

(pb) SELECT $C \cap S_k := C \cap P_k$.

(pi) As $C \cap I_k := \emptyset$ REMOVE nothing.

(pp) For $i \in C \cap S_k$ let $A^i_C := \{j \in B_k : (i,j) \in E(H_k)\}$. Order A^C_i totally via

$$j <_{A^i_C} \mathfrak{j} \Leftrightarrow (\xi_j <_{\xi_i} \xi_\mathfrak{j}) \vee (\xi_j = \xi_\mathfrak{j} \wedge j < \mathfrak{j}) \,. \tag{5.47}$$

Let $j_i := \min\{j \in A^i_C\}$ with respect to $<_{A^i_C}$. SELECT (i, j_i).

(pu) For $i \in C \cap S_k$ REMOVE all $(i,j) \in E(H_k)$ with $j_i \neq j \in A^i_C \subseteq B_k$.

(pc) REMOVE all of $E(C \cap S_k) \cap E(H_k)$.

We also modify the partition in algorithm 145 to

Algorithm 150 (modified Pen tree edge complement partition)**.** Let $\mathbb{T} \in \mathcal{T}_G$. Let L_k be the k^{th} level of $\mathbb{T}$. We partition $E \setminus E(\mathbb{T})$ into $\mathcal{A}_{PenM}(\mathbb{T}) \uplus \mathcal{C}_{PenM}(\mathbb{T})$. Let $0 \leq k \leq l$, $j \in L_k$, $i \in L_l$ and $e := (i,j) \in E \setminus E(\mathbb{T})$. Let

$$A^i_\mathbb{T} := \{j \in L_k : (i,j) \in E\} \neq \emptyset$$

and define a total order $<_{A^i_\mathbb{T}}$ on $A^i_\mathbb{T}$ via

$$j <_{A^i_\mathbb{T}} \mathfrak{j} \Leftrightarrow (\xi_j <_{\xi_i} \xi_\mathfrak{j}) \vee (\xi_j = \xi_\mathfrak{j} \wedge j < \mathfrak{j}) \,.$$

Then $e \in \mathcal{C}_{PenM}(\mathbb{T})$ iff one of the mutually exclusive conditions (5.48) holds:

$$l \geq k+2 \,, \tag{5.48a}$$
$$l = k+1 \,\wedge\, j <_{A^i_\mathbb{T}} \mathfrak{p}(i) \,. \tag{5.48b}$$

And $e \in \mathcal{A}_{PenM}(\mathbb{T})$ iff one of the mutually exclusive conditions (5.39) holds:

$$k = l \,, \tag{5.49a}$$
$$l = k+1 \,\wedge\, j >_{A^i_\mathbb{T}} \mathfrak{p}(i) \,. \tag{5.49b}$$

The modified scheme $PenM$ allows the formulation of a new restriction. If $\mathbb{T} \in \mathcal{T}_{G(\vec{\xi})}$, then

$$\forall\, k \in \mathbb{N}, i \in L_k, j \in L_{k-1} : \quad \xi_j <_{\xi_i} \xi_{\mathfrak{p}(i)} \Rightarrow (i,j) \in \mathcal{C}_{PenM}(\mathbb{T}) \,. \tag{5.50}$$

In particular, this holds for $\mathbb{T} \in \mathcal{T}_{PenM}(G(\vec{\xi}))$, in addition to the properties enounced in proposition 147.

5.6 Combining escaping and tree-operators

5.6.1 The motivation

We have seen in section 5.5.3 how Fernández & Procacci improved Dobrushin's condition by incorporating information about $\mathcal{I}^\star(\gamma)$. Condition (5.18b) called for a similar extension. It became quickly clear that *Pen* would not suffice. A partition scheme, which never branches to all of $\mathcal{I}^\star(\gamma)$, was needed.

The holy grail would be a scheme excluding a globally fixed neighbour of each vertex. This is impossible, though. The counterexample is a polymer system $(\mathcal{P}, \approx)$ isomorph (ignoring loops) to a large circle of size N. Take the cluster formed by $\vec{\xi} \in \mathcal{P}^N$ with $\operatorname{supp} \vec{\xi} = \mathcal{P}$, which is again isomorph to the circle of size N. It has N spanning trees and only one of them can be expanded to the full circle. This means, that every partition scheme has $N-1$ singleton trees on $G(\vec{\xi})$. Fix the root and the globally forbidden neighbours. Then every finite family approximation not returning to forbidden neighbours contains at most one of those $N-1$ trees. Hence no partition scheme with globally excluded neighbours exists.

Thus the search turned to guaranteeing the *needed exclusion locally* for the singleton trees of the partition scheme. This inspired the formalization of inductive partition schemes in section 5.4.4 as a framework for the *returning partition scheme* in section 5.6.2.

5.6.2 Returning (or self-avoiding) scheme

Let I be a finite and totally ordered set. Let $\vec{\xi} \in \mathcal{P}^I$ and $G := G(\vec{\xi})$. Assume that G is connected, that is it is a cluster. We present an inductive partition scheme *Ret* adapted to clusters. The static information comprises the total order on I, the cluster structure of G given by $\vec{\xi}$ and the choice of the root $o \in I$.

Algorithm 151 (*Ret* exploration)**.** Let $H \in \mathcal{C}_G$. For every k let H_k, T_k, U_k, B_k and P_k be as in algorithm 135. The missing parts to construct H_{k+1} from H_k are:

Call an edge $(i,j) \in E(C \cap P_k, B_k) \cap E(H)$ *same* (or **S**) iff $\xi_i = \xi_j$ and *different* (or **D**) iff $\xi_i \neq \xi_j$. Likewise call a vertex $i \in P_k$ same iff all such (i,j) are same and different iff there exists such a non-same (i,j). Finally we say that a connected component C of $H_k|_{U_k}$ is same iff all vertices in $C \cap P_k$ are same and different iff $C \cap P_k$ contains at least one different vertex.

If C is an **S** connected component of $H_k|_{U_k}$:

(rbs) SELECT $C \cap S_k := C \cap P_k$.

(ris) As $C \cap I_k = \emptyset$ REMOVE nothing.

(rps) For $i \in C \cap S_k$ let $j_i := \operatorname{argmin} \{j \in B_k : (i,j) \in E(H_k)\}$. SELECT (i, j_i).

(rus) For $i \in C \cap S_k$ REMOVE all $(i,j) \in E(H_k)$ with $j_i \neq j \in B_k$.

(rcs) REMOVE all of $E(C \cap S_k) \cap E(H_k)$.

If C is a **D** connected component of $H_k|_{U_k}$:

(rbd) SELECT $C \cap S_k := \{i \in C \cap P_k : i \text{ is } \mathbf{D}\}$.

(rid) $C \cap I_k = \{i \in C \cap P_k : i \text{ is } \mathbf{S}\}$. REMOVE all of $E(B_k, (C \cap I_k)) \cap E(H_k)$.

(rpd) For $i \in C \cap S_k$ let $j_i := \operatorname{argmin}\{j \in B_k : (i,j) \in E(H_k) \text{ is } \mathbf{D}\}$. SELECT (i, j_i).

(rudd) For $i \in C \cap S_k$ REMOVE every **D** $(i,j) \in E(H_k)$ with $j_i \neq j \in B_k$.

(ruds) For $i \in C \cap S_k$ REMOVE every **S** $(i,j) \in E(C \cap S_k, B_k) \cap E(H_k)$.

(rcd) REMOVE all of $E(C \cap S_k) \cap E(H_k)$.

Proposition 152. *The function $\mathcal{E}_{Ret}$ described in algorithm 151 is $\mathcal{C}_G \to \mathcal{T}_G$ and $\mathcal{E}_{Ret}|_{\mathcal{T}_G} = id_{\mathcal{T}_G}$.*

Proof. Follows from proposition 137. □

Algorithm 153 (*Ret* tree edge complement partition)**.** Let $\mathbb{T} \in \mathcal{T}_G$. Let L_k be the k^{th} level of $\mathbb{T}$. First we determine if an edge $(\mathfrak{p}(i), i)$ is a same or non-same edge:

$$s: \quad I \setminus \{o\} \to \{\mathbf{S}, \mathbf{D}\} \quad i \mapsto \begin{cases} \mathbf{S} & \xi_i = \xi_{\mathfrak{p}(i)} \\ \mathbf{D} & \xi_i \neq \xi_{\mathfrak{p}(i)} \, . \end{cases} \tag{5.51}$$

For $k \geq 1$ define the *equivalence relation* $\sim_{(k)}$ on L_k by

$$i \sim_{(k)} j \Leftrightarrow s(P(o,i) \setminus \{o\}) = s(P(o,j) \setminus \{o\}) \, , \tag{5.52}$$

where the equality on the rhs is taken in $\{\mathbf{S}, \mathbf{D}\}^k$. This implies that an equivalence class consists of either only same or only non-same nodes and whence we can extend s to them. For completeness let $\sim_{(0)}$ be the trivial equivalence relation on $L_0 = \{o\}$. There is a *tree structure* consistent with $\mathbb{T}$ on the equivalence classes, namely:

$$i \sim_{(k+1)} j \Rightarrow \mathfrak{p}(i) \sim_{(k)} \mathfrak{p}(j) \, , \tag{5.53}$$

that is equivalent vertices in L_{k+1} have equivalent parents in L_k. We therefore call $[\mathfrak{p}(i)]_{(k)}$ the *parent class* of $[i]_{(k+1)}$.

We partition $E \setminus E(\mathbb{T})$ into $\mathcal{A}_{Ret}(\mathbb{T}) \uplus \mathcal{C}_{Ret}(\mathbb{T})$. Let $0 \leq k \leq l$, $j \in L_k$, $i \in L_l$ and $e := (i,j) \in E \setminus E(\mathbb{T})$. Then $e \in \mathcal{C}_{Ret}(\mathbb{T})$ iff one of the mutually exclusive conditions (5.54) holds:

$$[j]_{(k)} \notin P([o]_{(0)}, [i]_{(l)}) \, , \tag{5.54a}$$
$$l \geq 2 \wedge [j]_{(k)} \in P([o]_{(0)}, [\mathfrak{p}(\mathfrak{p}(i))]_{(l-2)}) \wedge \xi_i \neq \xi_j \, , \tag{5.54b}$$
$$l \geq 2 \wedge [j]_{(k)} \in P([o]_{(0)}, [\mathfrak{p}(\mathfrak{p}(i))]_{(l-2)}) \wedge \xi_i = \xi_j \wedge s(C) = \mathbf{S} \, , \tag{5.54c}$$

where $C \in P([j]_{(k)}, [\mathfrak{p}(i)]_{(l-1)})$ the unique class with $\mathfrak{p}(C) = [j]_{(k)}$,

$$l \geq 1 \wedge [j]_{(k)} = [\mathfrak{p}(i)]_{(l-1)} \wedge \xi_i \neq \xi_j \wedge s(i) = \mathbf{D} \wedge j < \mathfrak{p}(i) \, , \tag{5.54d}$$
$$l \geq 1 \wedge [j]_{(k)} = [\mathfrak{p}(i)]_{(l-1)} \wedge \xi_i \neq \xi_j \wedge s(i) = \mathbf{S} \, , \tag{5.54e}$$

$$l \geq 1 \wedge [j]_{(k)} = [\mathfrak{p}(i)]_{(l-1)} \wedge \xi_i = \xi_j \wedge s(i) = \mathbf{S} \wedge j < \mathfrak{p}(i)\,. \tag{5.54f}$$

And $e \in \mathcal{A}_{Ret}(\mathbb{T})$ iff one of the mutually exclusive conditions (5.55) holds:

$$[j]_{(k)} = [i]_{(l)}\,, \tag{5.55a}$$

$$l \geq 2 \wedge [j]_{(k)} \in P([o]_{(0)}, [\mathfrak{p}(\mathfrak{p}(i))]_{(l-2)}) \wedge \xi_i = \xi_j \wedge s(C) = \mathbf{D}\,, \tag{5.55b}$$

where $C \in P([j]_{(k)}, [\mathfrak{p}(i)]_{(l-1)})$ the unique class with $\mathfrak{p}(C) = [j]_{(k)}$,

$$l \geq 1 \wedge [j]_{(k)} = \mathfrak{p}([i]_{(l)}) \wedge \xi_i \neq \xi_j \wedge s(i) = \mathbf{D} \wedge j > \mathfrak{p}(i)\,, \tag{5.55c}$$

$$l \geq 1 \wedge [j]_{(k)} = \mathfrak{p}([i]_{(l)}) \wedge \xi_i = \xi_j \wedge s(i) = \mathbf{D}\,, \tag{5.55d}$$

$$l \geq 1 \wedge [j]_{(k)} = \mathfrak{p}([i]_{(l)}) \wedge \xi_i = \xi_j \wedge s(i) = \mathbf{S} \wedge j > \mathfrak{p}(i)\,. \tag{5.55e}$$

Remark. Of particular importance are the *deep edges* in (5.55b). This is where we use the particular structure of $G(\vec{\xi})$. Paths in the tree returning to a previously visited polymer find here always an admissible edge to add, thus excluding the tree from $\mathcal{T}_{Ret}(G(\vec{\xi}))$.

Proposition 154. *The map*

$$Ret: \quad \mathcal{T}_G \to \mathcal{C}_G \quad \mathbb{T} \mapsto Ret(\mathbb{T}) := (I, E(\mathbb{T}) \uplus \mathcal{A}_{Ret}(\mathbb{T})) \tag{5.56}$$

is a partition scheme of G with $[\mathbb{T}, Ret(\mathbb{T})] = \mathcal{E}_{Ret}^{-1}(\mathbb{T})$.

Proof. If we admit that the partition in algorithm 153 satisfies (5.33), then proposition 154 is a direct consequence of proposition 138. Thus we show (5.33) for $\mathcal{E}_{Ret}$ and $\mathcal{A}_{Ret}(\mathbb{T})$.

Fix $\mathbb{T} \in \mathcal{T}_G$ and $H \in \mathcal{E}_{Ret}^{-1}(\mathbb{T})$. Let $0 \leq k \leq l$, $j \in L_k$, $i \in L_l$ and $e := (i,j) \in E \setminus E(\mathbb{T})$. Let $(H_n)_{n\in\mathbb{N}_0}$ and $(T_n)_{n\in\mathbb{N}_0}$ be the sequences associated with H from algorithm 151. For $\emptyset \neq F \subseteq E \setminus E(H)$ set $\tilde{H} := (I, E(H) \uplus F)$ and let $(\tilde{H}_n)_{n\in\mathbb{N}_0}$ and $(\tilde{T}_n)_{n\in\mathbb{N}_0}$ be its associated sequences from algorithm 151.

We define the *influence level* $m(e)$ of e as the unique solution of $[j]_{(k)} \curlywedge [i]_{(l)} \subseteq L_{m(e)}$. We claim that for each $N \in \mathbb{N}$ (compare with (5.41)):

$$(\forall\, e \in F: \, m(e) \geq N) \;\Rightarrow\; (H_n|_{T_n})_{n=0}^N = (\tilde{H}_n|_{\tilde{T}_n})_{n=0}^N\,. \tag{5.57}$$

To show (5.57) we proceed by induction over n, for $n \in [N]_0$. By definition $H_0|_{T_0} = \tilde{H}_0|_{\tilde{T}_0}$. For the induction step from $n < N$ to $n+1$ we show that algorithm 151 is not influenced by the presence of such an $e \in F$. The addition of e does not change $\tilde{P}_n$ nor the s classification of its vertices compared to P_n. Let $\mathfrak{i}$ and $\mathfrak{j}$ be the ancestors of i and j at level $n+1$ of $\mathbb{T}$ respectively. Then they are both in P_n and $\tilde{P}_n$. As $[\mathfrak{i}]_{(n+1)} = [\mathfrak{j}]_{(n+1)}$ in $\mathbb{T}$ we have two possibilities in $H_n|_{U_n}$: both $\mathfrak{i}$ and $\mathfrak{j}$ are classified $\mathbf{S}$ and in an $\mathbf{S}$ connected component of $H_n|_{U_n}$ or both $\mathfrak{i}$ and $\mathfrak{j}$ are classified $\mathbf{D}$ and in a $\mathbf{D}$ connected component of $H_n|_{U_n}$. In both cases the presence of e in $\tilde{H}$ could merge the connected components of i and j in $H_n|_{U_n}$ respectively into one connected component of $\tilde{H}_n|_{\tilde{U}_n}$, but only of the same classification. Therefore all vertices in $\tilde{P}_n = P_n$ end in connected components of $H_n|_{U_n}$ and $\tilde{H}_n|_{\tilde{U}_n}$ of the same classification respectively. Thus $S_n = \tilde{S_n}$ and $T_n = \tilde{T}_n$. Finally the selection of the parent in **(rbs)** and **(rbd)** is independent of $H_n|_{U_n}$ and $\tilde{H}_n|_{\tilde{U}_n}$ in all possible combinations. We conclude that $H_{n+1}|_{T_{n+1}} = \tilde{H}_{n+1}|_{\tilde{T}_{n+1}}$.

Observe that if $e \in \mathcal{A}_{Ret}(\mathbb{T})$, then by (5.55) e has influence level k. To show (5.33a) we assume that $F \subseteq \mathcal{A}_{Ret}(\mathbb{T})$. If e is of type (5.55a), then it is removed during iteration $(k-1)$ by **(rcs)** or **(rcd)**. If e is of type (5.55b), then it is removed by **(rid)** during iteration k. If e is of type (5.55c), then it is removed by **(rudd)** during iteration k. If e is of type (5.55e), then it is removed by **(rus)** during iteration k. If e is of type (5.55d), then it is removed by **(ruds)** during iteration k.

To show (5.33b) we assume that $F' := F \cap \mathcal{C}_{Pen}(\mathbb{T}) \neq \emptyset$. Choose $e \in F$ with $m(e) = N := \min\{m(f) : f \in F'\}$ minimal. We demonstrate that the presence of e causes $(H_n|_{T_n})_{n\in\mathbb{N}_0}$ to diverge from $(\tilde{H}_n|_{\tilde{T}_n})_{n\in\mathbb{N}_0}$ exactly at level $N+1$, that is (5.57) holds and $H_{N+1}|_{T_{N+1}} \neq \tilde{H}_{N+1}|_{\tilde{T}_{N+1}}$. We go through all the cases of (5.54):

Case e of type (5.54a): It is evident that $N < k \vee l$. Hence there exist ancestors $\mathfrak{i}$ and $\mathfrak{j}$ of i and j in L_{N+1} respectively with $[\mathfrak{p}(\mathfrak{i})]_{(N)} = [\mathfrak{p}(\mathfrak{j})]_{(N)} = [j]_{(k)} \curlywedge [i]_{(l)} \subseteq L_N$ ($i = \mathfrak{i}$ iff $l = N$ possible. Same for j and $\mathfrak{j}$). It also follows from (5.54a) that $[\mathfrak{i}]_{(N+1)} \neq [\mathfrak{i}]_{(N+1)}$. This means that during iteration N of algorithm 151, without loss of generality in that order, $\mathfrak{i}$ is classified as **S** in an **S** connected component of $H_N|_{U_N}$ and $\mathfrak{i}$ is classified as **D** in a **D** connected component of $H_N|_{U_N}$. The addition of e in $\tilde{H}$ places $\mathfrak{i}$ and $\mathfrak{j}$ in the same connected component of $\tilde{H}_N|_{\tilde{U}_N}$, via the path $\mathfrak{i} \leftrightarrow i \leftrightarrow j \leftrightarrow \mathfrak{j}$. This means that $\mathfrak{i}$ is an **S** vertex in a **D** connected component of $\tilde{H}_N|_{\tilde{U}_N}$ and thus is not selected into $\tilde{S_N}$ by **(rbd)**. Thus $T_{N+1} \neq \tilde{T}_{N+1}$.

Case e of type (5.54b): Here $N = k$. Let $\mathfrak{i}$ be the ancestor of i with $\mathfrak{p}([\mathfrak{i}]_{(N+1)}) = [j]_{(N)}$. As $\xi_i \neq \xi_j$ the addition of e in $\tilde{H}$ classifies i as **D** in step N. Therefore $i \in \tilde{S}_N$ by **(rbd)**, but $i \notin T_N$. Thus $T_{N+1} \neq \tilde{T}_{N+1}$.

Case e of type (5.54c): Here $N = k$. Let $\mathfrak{i}$ be the ancestor of i with $\mathfrak{p}([\mathfrak{i}]_{(N+1)}) = [j]_{(N)}$. As $\xi_i = \xi_j$ the addition of e in $\tilde{H}$ classifies i as **S** in step N. As $i \leftrightarrow \mathfrak{i}$ in $H_N|_{U_N}$ and hence $\tilde{H}_N|_{\tilde{U}_N}$ we know that i is in a **D** connected component of $\tilde{H}_N|_{\tilde{U}_N}$, namely the one of $\mathfrak{i}$. Therefore $i \in \tilde{S}_N$ by **(rbs)**, but $i \notin T_N$. Thus $T_{N+1} \neq \tilde{T}_{N+1}$.

Case e of type (5.54d): Here $N = k$. Let $\mathfrak{i}$ be the parent of i in $[j]_{(N)}$. The addition of e in $\tilde{H}$ lets **(rpd)** select j to be the parent of i in $\tilde{T}_{N+1}|_{\tilde{T}_{N+1}}$. Thus $T_{N+1} \neq \tilde{T}_{N+1}$.

Case e of type (5.54e): Here $N = k$. Let $\mathfrak{i}$ be the parent of i in $[j]_{(N)}$. The addition of e in $\tilde{H}$ classifies i as **D** during step N instead of **S**. Therefore its parent in $\tilde{T}_{N+1}|_{\tilde{T}_{N+1}}$ is chosen by **(rpd)** instead of **(rps)** and is not $\mathfrak{i}$ any more. Thus $T_{N+1} \neq \tilde{T}_{N+1}$.

Case e of type (5.54f): Here $N = k$. Let $\mathfrak{i}$ be the parent of i in $[j]_{(N)}$. The addition of e in $\tilde{H}$ lets **(rps)** select j to be the parent of i in $\tilde{T}_{N+1}|_{\tilde{T}_{N+1}}$. Thus $T_{N+1} \neq \tilde{T}_{N+1}$. □

Proposition 155 (Properties of $\mathcal{T}_{Ret}(G(\vec{\xi}))$). *Let $\mathbb{T} \in \mathcal{T}_{Ret}(G(\vec{\xi}))$ and let C_i be the set of children of i in $\mathbb{T}$. Then*

$$|C_i| = |\operatorname{supp} \vec{\xi}_{C_i}| \tag{5.58a}$$

$$(\operatorname{supp} \vec{\xi}_{C_i}) \setminus \{\xi_i\} \text{ is an independent subset of } \mathcal{I}^\star(\xi_i) \tag{5.58b}$$

$$\forall\, k \in \mathbb{N}_0, i \in L_k : \quad \operatorname{supp} \vec{\xi}_{[i]_{(k)}} \text{ is an independent subset of } \mathcal{P} \tag{5.58c}$$

$$\forall\, i \in I \setminus \{o\} : \quad i \text{ is } \mathbf{D} \Rightarrow \xi_i \notin \operatorname{supp} \vec{\xi}_{P(o,\mathfrak{p}(i))}\,. \tag{5.58d}$$

Proof. Fix k and let $i, j \in L_{k+1}$ with $i \sim_{(k+1)} j$. If $\xi_i \approx \xi_j$, then $e := (i, j) \in E$ as $G = G(\vec{\xi})$ (5.4b). Hence by (5.55a) $e \in E(Ret\mathbb{T})$ and thus $Ret(\mathbb{T}) \neq \mathbb{T}$. This shows (5.58c), which implies (5.58b), which in turn implies (5.58a).

Suppose there exists a vertex i violating (5.58d). If $\mathfrak{p}(i) = o$ we have a direct contradiction to the $\mathbf{D}$ classification of i. If $\mathfrak{p}(i) \neq o$, then the violation implies the existence of $j, \mathfrak{j} \in P(o, \mathfrak{p}(i))$ with $d_{\mathbb{T}}(o, j) + 1 = d_{\mathbb{T}}(o, \mathfrak{j}) < d_{\mathbb{T}}(o, i)$ and $\xi_i = \xi_j \neq \xi_{\mathfrak{j}}$. Take such a j minimal with respect to $d_{\mathbb{T}}(o, j)$. Consider the edge $e := (j, i)$, which is in E due to the fact that $G = G(\vec{\xi})$ (5.4b). It is admissible with respect to $\mathbb{T}$ of type (5.55b). Hence $e \in E(Ret(\mathbb{T}))$ and thus $Ret(\mathbb{T}) \neq \mathbb{T}$. □

5.6.3 Proof of proposition 126

Remark. Compare this proposition with theorem 148. It would be interesting to equip $\mathcal{L}$ with a *directed incompatibility relation* derived from the one on $\mathcal{P}$ and create a partition scheme on clusters of this new polymer system yielding the same result. In this case clusters should have structural properties inherited from $\mathcal{P}$ via $\mathcal{L}$. Of course this would probably demand directed partition schemes (that is on directed graphs) etc.

Proof of proposition 126. Let $\mathbb{T} \in \mathcal{T}_{Ret}(G(\vec{\xi}))$. Property (5.58d) is equivalent to:

$$\forall\, i \in I \setminus \{o\} : \nexists\, j, \mathfrak{j} \in P(o, \mathfrak{p}(i)) : \quad d_{\mathbb{T}}(o, j) < d_{\mathbb{T}}(o, \mathfrak{j}) < d_{\mathbb{T}}(o, i) \wedge \xi_i = \xi_j \neq \xi_{\mathfrak{j}}\,.$$

Therefore the polymer labels of $P(o, i)$ form a *lazy self-avoiding walk* on $\operatorname{supp} \vec{\xi}$. In particular we can look at the ordered sequence of *avoided polymers* for $i \in I$. By (5.21) we only regard $\vec{\xi}$ with $\operatorname{supp} \vec{\xi} \subseteq \mathcal{P} \setminus \{\varepsilon\}$. Hence the avoided polymers for i are a polymer-valued sequence starting with ε and adding ξ_j if we use a $\mathbf{D}$ edge after j on the path $P(o, i)$. The avoided polymers of o are (ε) and therefore the sequence is non-empty for every i. Call ε_i the last polymer in the avoided-polymer sequence of vertex i. We have $\xi_i \approx \varepsilon_i$ by construction for every $i \in I$. Hence $(\xi_i, \varepsilon_i)_{i \in I} \in \mathcal{L}^I$.

We focus on a vertex i and its children in $\mathbb{T}$. Property (5.58b) implies that their polymer labels form a compatible set. And property (5.58d) implies, that if we have an **S** i then $(\xi_i, \varepsilon_i) = (\xi_{\mathfrak{p}(i)}, \varepsilon_{\mathfrak{p}(i)})$, while if we have a **D** i then $\varepsilon_i = \xi_{\mathfrak{p}(i)}$. These constraints are determined by the extended labels $(\xi_i, \varepsilon_i)_{i\in I}$. Set $I := [n]_0$ with $o := n$. We drop all other constraints on $\mathbb{T}$, apply Penrose's identity (5.31) and get

$$\begin{aligned}
&|\mathfrak{u}(\vec{\xi})| \\
&= |\mathcal{T}_{Ret}(G(\vec{\xi}))| \\
&= \sum_{\tau\in\mathcal{T}_n} [\tau \in \mathcal{T}_{Ret}(G(\vec{\xi}))] \qquad (5.59a)\\
&\leq \sum_{\tau\in\mathcal{T}_n} \prod_{i\in[n]_0} c^{\text{multi}}_{s_i}((\xi_i,\varepsilon_i),(\xi_{i_1},\varepsilon_{i_1}),\dots,(\xi_{i_{s_i}},\varepsilon_{i_{s_i}}))\,.
\end{aligned}$$

where the $(s_i)_{i=0}^n$ denote the number of children of i in τ and

$$\begin{aligned}
&c^{\text{multi}}_n((\xi_0,\varepsilon_0),\dots,(\xi_n,\varepsilon_n)) \\
&\quad := \sum_{A\subseteq[n]} \left(\prod_{i\in A} [(\xi_i,\varepsilon_i) = (\xi_0,\varepsilon_0)] \prod_{i\neq j\in A} [\xi_j \not\approx \xi_i] \right) \\
&\qquad \times \left(\prod_{i\in[n]\setminus A} [\xi_i \in \mathcal{I}^\star(\xi_0)\setminus\{\varepsilon_0\}, \varepsilon_i = \xi_0] \prod_{i\neq j\in[n]\setminus A} [\xi_j \not\approx \xi_i] \right). \qquad (5.59b)
\end{aligned}$$

Let $Y := [0,\infty]^{\mathcal{L}}$. Denote by i_m the injection from X to Y *multiplexing* values by ignoring the escape coordinate in $\mathcal{L}$. Define the operator $\varphi^{\text{mult}} : Y \to Y$ by

$$\varphi^{\text{mult}}_{(\gamma,\varepsilon)}(\vec{u}) := \sum_{n\geq0} \frac{1}{n!} \sum_{(\vec{\xi},\vec{\varepsilon})\in\{(\gamma,\varepsilon)\}\times\mathcal{L}^n} c^{\text{multi}}_n(\vec{\xi},\vec{\varepsilon}) \prod_{i=1}^n u_{(\xi_i,\varepsilon_i)}\,.$$

We see that $c^{\text{multi}}_n(\vec{\xi},\vec{\varepsilon})$ is the coefficient of the product of the following terms:

$$\begin{aligned}
&(1+u_{(\gamma,\varepsilon)}) \\
&\quad = \sum_{n\geq0}\frac{1}{n!} \sum_{(\vec{\xi},\vec{\varepsilon})\in\{(\gamma,\varepsilon)\}\times\mathcal{L}^n} \prod_{i=1}^n [(\xi_i,\varepsilon_i) = (\gamma,\varepsilon)] \prod_{i\neq j=1}^n [\xi_j\not\approx\xi_i] \prod_{i=1}^n u_{(\xi_i,\varepsilon_i)}
\end{aligned}$$

and

$$\begin{aligned}
&\Xi_{\mathcal{I}^\star(\gamma)\setminus\{\varepsilon\}}(\vec{u}_{\{(\xi,\gamma):\,\xi\in\mathcal{I}^\star(\gamma)\setminus\{\varepsilon\}\}}) \\
&= \sum_{n\geq0}\frac{1}{n!} \sum_{(\vec{\xi},\vec{\varepsilon})\in\{(\gamma,\varepsilon)\}\times\mathcal{L}^n} \prod_{i=1}^n [\xi_i\in\mathcal{I}^\star(\gamma)\setminus\{\varepsilon\}, \varepsilon_i=\gamma] \prod_{i\neq j=1}^n [\xi_j\not\approx\xi_i] \prod_{i=1}^n u_{(\xi_i,\varepsilon_i)}\,.
\end{aligned}$$

Therefore

$$\varphi^{\text{mult}}_{(\gamma,\varepsilon)}(\vec{u}) = (1+u_{(\gamma,\varepsilon)})\,\Xi_{\mathcal{I}^\star(\gamma)\setminus\{\varepsilon\}}(\vec{u}_{\{(\xi,\gamma):\,\xi\in\mathcal{I}^\star(\gamma)\setminus\{\varepsilon\}\}})\,.$$

For $\vec{r}\in Y$ define the tree operator

$$S_{\vec{r}}: \quad Y\to Y \quad \vec{u}\mapsto \vec{r}\varphi^{\text{mult}}(\vec{u})\,.$$

Using the fact that

$$\forall \gamma \in \mathcal{P}: \quad \varphi_\gamma^{\mathrm{cTc}}(\vec{\mu}) = \max\{\varphi_{(\gamma,\varepsilon)}^{\mathrm{mult}}(i_m(\vec{\mu})) : \varepsilon \in \mathcal{I}^\star(\gamma)\}$$

we deduce from (5.22b) that

$$S_{i_m(\vec{\rho})}(i_m(\vec{\mu})) \leq i_m(\vec{\mu})\,.$$

We apply proposition 139 for $k = 1$ to $S_{i_m(\vec{\rho})}$ and get a finite series fixpoint $R(i_m(\vec{\rho})) \leq i_m(\vec{\mu})$. Then (5.22c) follows from (5.59a):

$$\forall (\gamma,\varepsilon) \in \mathcal{L}: \quad \rho_\gamma \Psi^\star_{(\gamma,\varepsilon)}(\vec{\rho}) \leq R_{(\gamma,\varepsilon)}(i_m(\vec{\rho})) \leq \mu_\gamma\,.$$

As in the proof of proposition 142 we see, that both $c_0((\gamma,\varepsilon)) = 1$ and $c_1((\gamma,\varepsilon),(\gamma,\varepsilon)) = 1$ as we count the clusters $G(\gamma)$ and $G(\gamma,\gamma)$. All terms are monotone increasing in $\vec{\rho}$. Hence we assume without loss of generality that $\vec{\rho} > 0$ or pass to a reduced polymer system. We apply propositions 140 and 141 to see that $\vec{\rho} < \vec{1}$ and

$$\forall (\gamma,\varepsilon) \in \mathcal{L}: \qquad \Psi^\star_{(\gamma,\varepsilon)}(\vec{\rho}) \leq \frac{\varphi_{(\gamma,\varepsilon)}^{\mathrm{mult}}(i_m(\vec{\mu})) - \mu_\gamma}{1-\rho_\gamma} \leq \frac{\widehat{\varphi_\gamma^{\mathrm{cTc}}}(\vec{\mu})}{1-\rho_\gamma}\,.$$

This proves (5.22c). Applying (5.3e) we get

$$\begin{aligned}\forall (\gamma,\varepsilon) \in \mathcal{L}: \qquad \Phi^\gamma_{\mathcal{P}\setminus\{\varepsilon\}}(-\vec{\rho}) &\geq \exp(-\rho_\gamma \Psi^\star_{(\gamma,\varepsilon)}(\vec{\rho})) \\ &\geq (1-\rho_\gamma)^{\varphi_{(\gamma,\varepsilon)}^{\mathrm{mult}}(i_m(\vec{\mu}))} \geq (1-\rho_\gamma)^{\widehat{\varphi_\gamma^{\mathrm{cTc}}}(\vec{\mu})}\,,\end{aligned}$$

whence (5.22d) holds. □

5.6.4 Relation between the pinned series

Sketch of proof of (5.23a). The idea is to decompose every tree according to the polymers labels of its first level. The relevant restriction for the first level is (5.58b). Note that there is no escape restriction, as long as we stay in the root polymer γ. Thus the subtrees at the child of 0 with polymer label γ have no restriction yet and give a Ψ_γ part. The other children form a compatible subset of $\mathcal{I}^\star(\gamma)$. This gives a product of factors $\Psi_{(\xi,\gamma)}$. Summing over all compatible subsets of $\mathcal{I}^\star(\gamma)$ yields $\Xi_{\mathcal{I}^\star(\gamma)}$. □

Rewrite (5.23a) into

$$\Psi_\gamma(\vec{\rho}) = \frac{\Xi_{\mathcal{I}^\star(\gamma)}\Big(i_m(\vec{\rho})\Psi^\star(i_m(\vec{\rho}))\Big)}{1-\rho_\gamma \Xi_{\mathcal{I}^\star(\gamma)}\Big(i_m(\vec{\rho})\Psi^\star(i_m(\vec{\rho}))\Big)}\,.$$

If $\rho_\gamma \Xi_{\mathcal{I}^\star(\gamma)}\Big(i_m(\vec{\rho})\Psi^\star(i_m(\vec{\rho}))\Big) < 1$, then this also constitutes a bound of $\Psi_\gamma(\vec{\rho})$.

5.6.5 Improvements

There are two possible extensions: keeping a longer history and respecting uncles.

Longer history: We can extend the label space to keep a longer history of length m around:

$$\mathcal{L} := \{(\xi_0, \dots, \xi_m) : \xi_0 \in \mathcal{P}, \forall\, i \in [m] : \xi_i \in \mathcal{I}^\star(\xi_{i-1}) \setminus \{\xi_0, \dots, \xi_{i-2}\}\} .$$

Again ξ_0 is the usual polymer label and $(\xi_1, \dots, \xi_m)$ a *self-avoiding path* of length m starting at ξ_0. This is only justified for pairs (Λ, γ) which are m-escaping (that is there is a length m self-avoiding path starting at γ in $\mathcal{P} \setminus \Lambda$).

A priori this means even less admissible pairs, but if $\mathcal{P}$ is infinite and connected one can order each finite Λ such that each escaping pair in its telescoping product (5.3a) is *m-escaping*.

This extension is complementary to the depth k approximations from section 5.4.6. The first one heeds more restrictions along the paths from the root, while the second one heeds more restrictions on the same level of the tree.

Respecting uncles: The same modification as in section 5.5.4 can be applied to exclude some of the possible uncles in the parent equivalence class of a **D** node. It only applies to approximations of depth 2 or more.

5.6.6 Proof of the optimality on trees

We claimed that (5.18b) is optimal on trees. We show this by proving the optimality of (5.22b) for polymer systems isomorph to a tree with loops. In this case (5.22b) reduces to (5.18b).

Proposition 156. *Let $(\mathcal{P}, \approx)$ be a tree with loops. Fix $(\gamma, \varepsilon) \in \mathcal{L}$. Let $\vec{\xi} \in \mathcal{P} \setminus \{\varepsilon\}^I$ with $\xi_o = \gamma$ and assume that $\operatorname{supp} \vec{\xi}$ is connected. We claim that*

$$\tau \in \mathcal{T}_{Ret}(G(\vec{\xi})) \Leftrightarrow \tau \in \mathcal{T}_I \text{ fulfils (5.58a), (5.58b) and (5.58d)} . \tag{5.60}$$

Proof. $\Rightarrow$ follows from (5.58). For $\Leftarrow$ let $\tau \in \mathcal{T}_I$ fulfil the rhs of (5.60). We show that $\tau \in \mathcal{T}_{G(\vec{\xi})}$ and $\mathcal{A}_{Ret}(\tau) = \emptyset$.

Property (5.58b) of τ implies that every edge $(i, j) \in E(\tau)$ has incompatible polymers at its endpoints. Thus $E(\tau) \subseteq E(G(\vec{\xi}))$ and $\tau \in \mathcal{T}_{G(\vec{\xi})}$.

Let $\mathcal{P}_\xi := \{\lambda \in \mathcal{P} : \xi \in \overline{\lambda\varepsilon}\}$. This way $(\mathcal{P}_\xi, \approx|_{\mathcal{P}_\xi})$ is a tree with loops and root ξ. Property (5.58d) implies, that once use a **D** edge from γ to ξ to enter a subtree, we can not use the label γ in the subtree any more. This amounts to restricting the available polymers labels for the subtree from $\mathcal{P}_\gamma$ to $\mathcal{P}_\xi$:

$$\forall\, i \in I : \quad \operatorname{supp}(\vec{\xi}_{V(\tau^i)}) \subseteq \mathcal{P}_{\xi_i} . \tag{5.61a}$$

It is at this point, where we use the tree-like shape of $\mathcal{P}$: forbidding γ restricts us to a subtree, for lack of other connections in $(\mathcal{P}, \approx)$. Combining (5.61a) with

(5.58a), that is every vertex can have at most one child with a given polymer label, results in

$$\forall \xi \in \operatorname{supp} \vec{\xi} : \exists j, i \in I : \quad C_\xi = \{ i \in I : \xi_i = \xi \} = P(o,i) \setminus (P(o,j) \setminus \{j\}) \,. \tag{5.61b}$$

That is, the vertices with the same polymer label are all along a finite downpath in the tree.

Let $0 \leq k \leq l$, $j \in L_k$, $i \in L_l$ and $e := (i,j) \in E \setminus E(\tau)$. We suppose that $(i,j) \in \mathcal{A}_{Ret}(\tau)$ and derive a contradiction in all cases of (5.55). Hence $\mathcal{A}_{Ret}(\tau) = \emptyset$.

Case (5.55a): We look at $s(P(o,i) \setminus \{o\}) = s(P(o,j) \setminus \{o\})$. If at some **D** step the paths $P(o,i)$ and $P(o,j)$ stepped to two different polymer labels, then (5.61a) asserts that $\xi_i \not\approx \xi_j$, as they are in disjoint, and thus compatible, polymer subsystems. Therefore $(i,j) \notin \mathcal{A}_{Ret}(\tau)$. If $\vec{\xi}_{P(o,i)} = \vec{\xi}_{P(o,j)}$, then some vertex on the path had two children with the same polymer label. This contradicts (5.58a).

Case (5.55b): The **D** step demanded for C in (5.55b) implies that $\xi_i \neq \xi_j$.

Case (5.55c): By (5.61a) all different uncles in $[\mathfrak{p}(i)]_{(l-1)}$ are in disjoint, and thus compatible, polymer subsystems.

Case (5.55d): By (5.61b) there are no same uncles.

Case (5.55e): By (5.61b) there are no same uncles. □

For $\tau \in \mathcal{T}_I$ extend the labels on $V(\mathbb{T}) = I$ to $(\xi_i, \varepsilon_i)_{i \in I}$, as in section 5.6.3. The claim (5.60) is equivalent to equality in the upper bound (5.59a):

$$|\mathfrak{u}(\vec{\xi})| = |\mathcal{T}_{Ret}(G(\vec{\xi}))| = \sum_{\tau \in \mathcal{T}_I} \prod_{i \in I} c^{\mathrm{multi}}_{s_i}((\xi_i, \varepsilon_i), (\xi_{i_1}, \varepsilon_{i_1}), \ldots, (\xi_{i_{s_i}}, \varepsilon_{i_{s_i}})) \,. \tag{5.62}$$

The chain of reasoning for the optimality goes as follows: (5.62) and (5.60) imply that for *Ret*, no improvement (more history, uncles, depth k, ...) can improve on the above depth 1 approximation. Therefore the derived depth 1 condition $\vec{\rho}\,\varphi^{\mathrm{cTc}}(\vec{\mu}) = \vec{\rho}\,\varphi^{\mathrm{esc}}(\vec{\mu}) \leq \vec{\mu}$ is asymptotically (depth k approximations with all conditions for $k \to \infty$) optimal. As every partition scheme has to give the same result in the $k \to \infty$ limit for optimal depth k approximations based on it, we have shown that φ^{esc} is optimal for tree-like polymer systems.

This section shows optimality in one particular case. At far more general question has been asked by Roberto Fernández on my last visit to Utrecht:

Question 157. Is (5.18b) the best condition obtainable from the fundamental identity (5.3c) *for every* polymer system?

5.7 Extensions and related models

5.7.1 The robustness of the condition by Kotecký & Preiss

Historically, the first condition was by Kotecký & Preiss [KP86]. It is

$$\forall \gamma \in \mathcal{P}: \quad \rho_\gamma \le \mu_\gamma \exp\left(- \sum_{\xi \in \mathcal{I}(\gamma)} \mu_\xi \right) . \tag{5.63}$$

In the tree-operator framework it is a consequence of theorem 119 after relaxing the coefficients c_n^{FP} (or c_n^{Dob}) to:

$$c_n^{\mathrm{KP}}(\vec{\xi}) := \prod_{i=1}^{n} [\xi_0 \approx \xi_i] . \tag{5.64}$$

This relaxation holds for *every* partition scheme on a cluster. That is, the relaxation is not a property of the choice of the partition scheme (*Pen* by Fernández & Procacci), but solely of the structural properties of the cluster $G(\vec{\xi})$. Summing up we get

$$\begin{aligned}
&\varphi_\gamma^{\mathrm{KP}}(\vec{\mu}) \\
&= \sum_{n\ge 0} \frac{1}{n!} \sum_{\vec{\xi} \in \mathcal{P}^n} c_n^{\mathrm{KP}}(\gamma, \vec{\xi}) \prod_{i=1}^{n} \mu_{\xi_i} \\
&= \sum_{n\ge 0} \frac{1}{n!} \sum_{\vec{\xi} \in \mathcal{I}(\gamma)^n} \prod_{i=1}^{n} \mu_{\xi_i} \\
&= \sum_{n\ge 0} \frac{1}{n!} \sum_{\substack{(M_\xi)_{\xi \in \mathcal{I}(\gamma)} \\ \biguplus_{\xi \in \mathcal{I}(\gamma)} M_\xi = [n]}} \prod_{\xi \in \mathcal{I}(\gamma)} (\mu_\xi)^{|M_\xi|} \\
&= \sum_{n\ge 0} \frac{1}{n!} \sum_{\substack{(m_\xi)_{\xi \in \mathcal{I}(\gamma)} \in \mathbb{N}_0^{\mathcal{I}(\gamma)} \\ \sum_{\xi \in \mathcal{I}(\gamma)} m_\xi = n}} \prod_{\xi \in \mathcal{I}(\gamma)} \frac{(\mu_\xi)^{m_\xi}}{m_\xi!} \\
&= \prod_{\xi \in \mathcal{I}(\gamma)} \left(\sum_{n \ge 0} \frac{\mu_\xi^n}{n!} \right) && \text{by (5.28c)} \\
&= \prod_{\xi \in \mathcal{I}(\gamma)} e^{\mu_\xi} \\
&= \exp\left(\sum_{\xi \in \mathcal{I}(\gamma)} \mu_\xi \right) .
\end{aligned}$$

5.7.2 Thinking about exactness

Call a polymer system $(\mathcal{P}, \approx)$ *escaping orderable* iff there is a global total order on $\mathcal{P}$ such that, for every $\Lambda \Subset \mathcal{P}$, $(\Lambda, \max \Lambda)$ is escaping.

Examples are trees and the k-fuzz of $\mathbb{Z}$. In my opinion this is intimately related to the question of exact calculation of critical points and exact representation of Shearer's measure. In these cases we have exact inductive arguments on the polymer level and the same should be true on the cluster level (compare with section 5.6.6).

Question 158. Are escaping orderable polymer systems identical with those not containing an infinite, connected subgraph quasi-isometric to $\mathbb{R}^d$, for some $d \geq 2$?

5.7.3 Uniting the greedy and escaping approach

The best classic condition (5.15b) by Fernández & Procacci and the best escaping condition (5.22b) are not comparable: there are examples of $(\mathcal{P}, \approx)$, where one beats the other. The difference is a trade-off between ignoring the smallest (with respect to $\vec{\mu}$ and thus $\vec{\rho}$) polymer in $\mathcal{I}^\star(\gamma)$ and the additional factor $(1+\mu_\gamma)$. Even for a given graph, it might be highly dependent of the choice of $\vec{\rho}$.

The easy way out is to take the log-convex hull of all points satisfying one of the two conditions.

A more sophisticated approach classifies each non-loop edge in $(\mathcal{P}, \approx)$, that is each $(\gamma, \varepsilon) \in \mathcal{L}$, as escaping or greedy. It then optimises over all such classifications. The aim would be to construct a partition scheme behaving greedily or self-avoiding, according to the edges it considers. The partition schemes *Pen* and *Ret* are extreme cases of such a classification. By taking the best classification, which might change with $\vec{\mu}/\vec{\rho}$, even for a given $(\mathcal{P}, \approx)$, one would get a combined condition improving on both (5.15b) and (5.22b). This combined condition is probably useless in practise: the usual case are conditions for quasi-transitive $(\mathcal{P}, \approx)$ with quasi-transitive fugacities. But it would fuse the two approaches and provide an automatically adapted balance between the two behaviours (greedy and self-avoiding).

5.7.4 Application to related models

The new condition (5.22b) should be transferable to the hard-sphere case [FPS07]. One just needs to introduce a second parameter $0 < r < R$, which describes "same-ness" of two points. Calculations are in progress.

Think about generalizations to other interactions as in [SS05], [Pro07] or [Far08]. Possible solutions to the soft-core interaction problem are cutoffs (dividing interactions into soft and hard parts), etc. The basic idea of a self-avoiding walk should be salvageable, though – as long as we still have hard-core self-repulsion.

5.8 Miscellaneous stuff

5.8.1 Minimal weight scheme

Choose $\mathbb{Q}$-linear independent weights $\vec{w} \in \mathbb{R}^E$. Let $\min : \mathcal{C}_G \to \mathcal{T}_G$ be the function choosing the minimal weight spanning tree of H with respect to $\vec{w}_{E(H)}$.

Note that the minimal weighted spanning tree is unique.

Proposition 159 ([SS05, lemma 2.2])**.** *The function $Min : \mathcal{T}_G \to \mathcal{C}_G$ defined by*

$$Min(T) := \left(V, \bigcup_{H:\, \min(H)=T} E(H) \right) \tag{5.65}$$

is a partition scheme.

Proposition 160 ([SS05, lemma 2.2])**.** *Let $\mathbb{T}_{\max} = \max(G)$ be the maximal weighted spanning tree of G with respect to $\vec{w}$. Then $Min(\mathbb{T}_{\max}) = \mathbb{T}_{\max}$ and $\mathbb{T}_{\max} \in \mathcal{T}_{Min}(G)$.*

Corollary 161. *For every G and partition scheme S of $\mathcal{C}_G$, we have $|\mathcal{T}_S(G)| \geq 1$.*

Remark. After [SS05, note on middle of page 12] ordering the edges $(e_1, \dots, e_n)$ and taking the weights $w_{e_n} := 2^n$ one gets the *lexicographical spanning tree.*

5.8.2 The polymer system has no strong dependency graph

For $\Lambda \Subset \mathcal{P}$ and $\vec{z} \in [0, \infty[^{\mathcal{P}}$ there is a *Boltzmann measure* $\lambda_{\Lambda,\vec{z}}$ on $\{0,1\}^\Lambda$ given by:

$$\lambda_{\Lambda,\vec{z}} : \quad \{0,1\}^\Lambda \to [0,1] \quad A \mapsto \begin{cases} \Xi_\Lambda(\vec{z})^{-1} \prod_{\xi \in A} z_\xi & \text{if } A \text{ compatible} \\ 0 & \text{else.} \end{cases} \tag{5.66}$$

Recall the notion of strong dependency graph: G is a strong dependency graph of the Bernoulli random field (short BPF) $Y := (Y_v)_{v \in V}$ iff for all $U, W \subseteq V$ with $d(U,W) > 1$ the subfield Y_U is independent of Y_W.

Proposition 162. *The BRF with distribution $\lambda_{\Lambda,\vec{z}}$ may not have $(\Lambda, \approx|_{\Lambda^2})$ as strong dependency graph.*

Proof. Choose $\Lambda := \{a, b, c\}$ with $a \approx b \approx c$, but $a \not\approx c$. Thus $(\Lambda, \approx|_{\Lambda^2})$ resembles a line of length 2 with loops at each polymer. Choose $z > 0$. Then $\Xi_\Lambda(z) = 1 + 3z + z^2$. Therefore

$$\lambda_{\Lambda,z}(1_a 1_c) = \frac{z^2}{\Xi_\Lambda(z)} \quad \text{and} \quad \lambda_{\Lambda,z}(1_a) = \frac{z + z^2}{\Xi_\Lambda(z)}\,.$$

We see, that

$$\lambda_{\Lambda,z}(1_a 1_c) \neq \lambda_{\Lambda,z}(1_a)\lambda_{\Lambda,z}(1_c)\,.$$

□

5.8.3 Proof of cluster expansion

Proof of (5.6a). This follows the exposition in [MS10]. We write the partition function as a sum over induced graphs and then factor them into their clusters. Hidden is the fact that taking the exponential of a generating function amounts to enumerating sequences of them: in our case we enumerate the clusters of the induced graphs.

Let $\vec{\xi} \in \mathcal{P}^n$. Then

$$\prod_{1\le i<j\le n} [\xi_i \not\approx \xi_j] = \prod_{1\le i<j\le n} (1 - [\xi_i \approx \xi_j]) = \sum_{A \subseteq E(G(\vec{\xi}))} \prod_{(i,j):=e\in A} (-1) = \sum_{A \subseteq E(G(\vec{\xi}))} (-1)^{|A|} = \sum_{H \in \mathcal{C}_{G(\vec{\xi})}} (-1)^{|E(H)|}\,.$$

The last quantity factorizes over connected components of $G(\vec{\xi})$. Let $(C_k)_{k=1}^m$ be the cluster partition of I with respect to $G(\vec{\xi})$. Then

$$\prod_{1\le i<j\le n} [\xi_i \not\approx \xi_j] = \prod_{k=1}^m \prod_{\substack{1\le i<j\le n\\ i,j\in C_k}} [\xi_i \not\approx \xi_j] = \prod_{k=1}^m \mathfrak{u}(\vec{\xi}_{C_k})\,.$$

We rewrite the partition function (5.1b) as

$$\begin{aligned}
&\Xi_\Lambda(\vec{z})\\
&= \sum_{n\ge 0} \frac{1}{n!} \sum_{\vec{\xi}\in\Lambda^n} \left(\prod_{1\le i<j\le n} [\xi_i \not\approx \xi_j] \right) \prod_{i=1}^n z_{\xi_i}\\
&= \sum_{n\ge 0} \frac{1}{n!} \sum_{\vec{\xi}\in\Lambda^n} \mathfrak{u}(\vec{\xi}) \prod_{i=1}^n z_{\xi_i}\\
&= \sum_{n\ge 0} \frac{1}{n!} \sum_{\vec{\xi}\in\Lambda^n} \left(\sum_{m\ge 0} \sum_{\substack{(C_k)_{k=1}^m\\ \biguplus_{k=1}^m C_k=[n]}} \prod_{k=1}^m \mathfrak{u}(\vec{\xi}_{C_k}) \right) \prod_{i=1}^n z_{\xi_i}\\
&= \sum_{n\ge 0} \frac{1}{n!} \sum_{\vec{\xi}\in\Lambda^n} \sum_{m\ge 0} \sum_{\substack{(C_k)_{k=1}^m\\ \biguplus_{k=1}^m C_k=[n]}} \prod_{k=1}^m \left(\mathfrak{u}(\vec{\xi}_{C_k}) \prod_{i\in C_k} z_{\xi_i} \right)\\
&= \sum_{n\ge 0} \frac{1}{n!} \sum_{m\ge 0} \sum_{\substack{(C_k)_{k=1}^m\\ \biguplus_{k=1}^m C_k=[n]}} \sum_{\vec{\xi}\in\Lambda^n} \prod_{k=1}^m \left(\mathfrak{u}(\vec{\xi}_{C_k}) \prod_{i\in C_k} z_{\xi_i} \right)\\
&= \sum_{n\ge 0} \frac{1}{n!} \sum_{m\ge 0} \sum_{\substack{(C_k)_{k=1}^m\\ \biguplus_{k=1}^m C_k=[n]}} \prod_{k=1}^m \left(\sum_{\vec{\xi}\in\Lambda^{C_k}} \mathfrak{u}(\vec{\xi}) \prod_{i\in C_k} z_{\xi_i} \right)\\
&= \sum_{n\ge 0} \frac{1}{n!} \sum_{m\ge 0} \sum_{\substack{(n_k)_{k=1}^m\\ \sum_{k=1}^m n_k=n}} \frac{n!}{m!} \prod_{k=1}^m \left(\frac{1}{n_k!} \sum_{\vec{\xi}\in\Lambda^{[n_k]}} \mathfrak{u}(\vec{\xi}) \prod_{i\in[n_k]} z_{\xi_i} \right)\\
&= \sum_{m\ge 0} \frac{1}{m!} \sum_{n\ge 0} \sum_{\substack{(n_k)_{k=1}^m\\ \sum_{k=1}^m n_k=n}} \prod_{k=1}^m \left(\frac{1}{n_k!} \sum_{\vec{\xi}\in\Lambda^{[n_k]}} \mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n_k} z_{\xi_i} \right)
\end{aligned}$$

$$= \sum_{m\geq 0} \frac{1}{m!} \left(\sum_{n\geq 0} \frac{1}{n!} \sum_{\vec{\xi}\in\Lambda^n} \mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} z_{\xi_i} \right)^m$$

$$= \exp \left(\sum_{n\geq 0} \frac{1}{n!} \sum_{\vec{\xi}\in\Lambda^n} \mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} z_{\xi_i} \right) .$$

□

5.8.4 Proof of worst case

Let $\vec{\xi} \in \mathcal{P}^n$ and $\vec{z} \in \mathbb{C}^n$. Then the alternating sign property (5.7) and the fact that $\mathrm{Re}(a) \leq |a|$ imply that

$$\mathrm{Re}\left(\mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} z_{\xi_i} \right) = -|\mathfrak{u}(\vec{\xi})| \,\mathrm{Re}\left(\prod_{i=1}^{n} (-z_{\xi_i}) \right)$$
$$\geq -|\mathfrak{u}(\vec{\xi})| \prod_{i=1}^{n} |z_{\xi_i}| = \mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} (-|z_{\xi_i}|) .$$

This way we show (5.8b)

$$\begin{aligned}
&\log \Xi_\Lambda(-|\vec{z}|) \\
&= \sum_{n\geq 1} \frac{1}{n!} \sum_{\vec{\xi}\in\Lambda^n} \left(\mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} |z_{\xi_i}| \right) && \text{by (5.6a)} \\
&\geq \sum_{n\geq 1} \frac{1}{n!} \sum_{\vec{\xi}\in\Lambda^n} \mathrm{Re}\left(\mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} z_{\xi_i} \right) \\
&= \mathrm{Re}\left(\sum_{n\geq 1} \frac{1}{n!} \sum_{\vec{\xi}\in\Lambda^n} \mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} z_{\xi_i} \right) \\
&= \mathrm{Re}\left(\log \Xi_\Lambda(\vec{z})\right) && \text{by (5.6a)} .
\end{aligned}$$

Also, if $\vec{\xi} \in \{\gamma\} \times \mathcal{P}^n$ and $\vec{z} \in \mathbb{C}^n$ the alternating sign property (5.7) implies that

$$\left| \mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} z_{\xi_i} \right| = |\mathfrak{u}(\vec{\xi})| \left| \prod_{i=1}^{n} (-z_{\xi_i}) \right| \leq |\mathfrak{u}(\vec{\xi})| \prod_{i=1}^{n} |z_{\xi_i})| = \mathfrak{u}(\vec{\xi}) \prod_{i=1}^{n} (-|z_{\xi_i})|) .$$

This proves (5.8d) and (5.8e).

5.8.5 Proof of monotonicity and shape properties

Proof of lemma 117. From (5.6d) and (5.7) we deduce that

$$\log \Phi^\gamma_\Lambda(-\vec{\rho}) = -\sum_{n\geq 1} \frac{1}{n!} \sum_{\vec{\xi}\in\Lambda^n} [\gamma \in \mathrm{supp}(\vec{\xi})] |\mathfrak{u}(\vec{\xi})| \prod_{i=1}^{n} \rho_{\xi_i}$$

Thus increasing ρ or Λ decreases the logarithm and thus $\Phi^{\gamma}_{\Lambda}(-\vec{\rho})$. The first increase is synonym with $\mathcal{R}_{\mathcal{P}}$ being a down-set. If we add incompatibilities to $\approx$, then more $\vec{\xi}$ have a connected $G(\vec{\xi})$, leading to the same result. Rewriting

$$\begin{aligned}\mathcal{R}_{\mathcal{P}} &= \left\{\vec{\rho} \in [0,\infty[^{\mathcal{P}} : \quad \forall\, \Lambda \Subset \mathcal{P} : \quad \Xi_{\Lambda}(-\vec{\rho}) > 0\right\} \\ &= \left\{\vec{\rho} \in [0,\infty[^{\mathcal{P}} : \quad \forall\, \gamma \in \Lambda \Subset \mathcal{P} : \quad \Phi^{\gamma}_{\Lambda}(-\vec{\rho}) > 0\right\}\end{aligned}$$

we see that more incompatibilities imply more restrictions on $\vec{\rho}$, whence $\mathcal{R}_{\mathcal{P}}$ can only decrease.

Remark. This proof for log-convexity follows the exposition [Kra01, sections 2.3 and 3.4.3], in particular the proof of [Kra01, proposition 2.3.15].

Let $\vec{\rho}, \vec{\mu} \in \mathcal{R}_{\mathcal{P}}$ and $\lambda \in [0,1]$. We recall *Young's inequality*:

$$\forall\, a, b \geq 0 : \quad a^{\lambda} b^{1-\lambda} \leq \lambda a + (1-\lambda) b\,.$$

We rewrite the cluster expansion (5.6a) using Penrose's identity (5.31) into a sum over multi-indices $\vec{n} \in (\mathbb{N}_0)^{\Lambda}$. As all summands are non-negative real values and $-\log \Xi_{\Lambda}(-\vec{\rho}) < \infty$ the rearrangement is well-defined.

$$-\log \Xi_{\Lambda}(-\vec{\rho}) = \sum_{n \geq 1} \frac{1}{n!} \sum_{\vec{\xi} \in \Lambda^n} [\gamma \in \operatorname{supp}(\vec{\xi})] |\mathfrak{u}(\vec{\xi})| \prod_{i=1}^{n} \rho_{\xi_i} = \sum_{\vec{n} : \Lambda \to \mathbb{N}_0} a_{\vec{n},\Lambda} \prod_{\xi \in \Lambda} \rho_{\xi}^{n_{\xi}}\,.$$

Then

$$\begin{aligned} &- \log \Xi_{\Lambda}(-\vec{\rho}^{\lambda} \vec{\mu}^{1-\lambda}) \\ &= \sum_{\vec{n} : \Lambda \to \mathbb{N}_0} a_{\vec{n},\Lambda} \prod_{\xi \in \Lambda} (\rho_{\xi}^{\lambda} \mu_{\xi}^{1-\lambda})^{n_{\xi}} \\ &= \sum_{\vec{n} : \Lambda \to \mathbb{N}_0} a_{\vec{n},\Lambda} \left(\prod_{\xi \in \Lambda} \rho_{\xi}^{n_{\xi}} \right)^{\lambda} \left(\prod_{\xi \in \Lambda} \mu_{\xi}^{n_{\xi}} \right)^{1-\lambda} \\ &\leq \sum_{\vec{n} : \Lambda \to \mathbb{N}_0} a_{\vec{n},\Lambda} \left(\lambda \prod_{\xi \in \Lambda} \rho_{\xi}^{n_{\xi}} + (1-\lambda) \prod_{\xi \in \Lambda} \mu_{\xi}^{n_{\xi}} \right) \\ &= -\lambda \log \Xi_{\Lambda}(-\vec{\rho}) - (1-\lambda) \log \Xi_{\Lambda}(-\vec{\mu})\,. \end{aligned}$$

Thus $\vec{\rho}^{\lambda} \vec{\mu}^{1-\lambda} \in \mathcal{R}_{\mathcal{P}}$ and

$$\Xi_{\Lambda}(-\vec{\rho}^{\lambda} \vec{\mu}^{1-\lambda}) \geq \Xi_{\Lambda}(-\vec{\rho})^{\lambda} \Xi_{\Lambda}(-\vec{\mu})^{1-\lambda}\,.$$

□

Proposition 163. *If $\vec{0} \leq \vec{\rho} \leq \vec{\mu} \in \mathcal{R}_{\mathcal{P}}$, then*

$$\forall\, (\Lambda, \gamma) : \quad 1 \geq \Phi^{\gamma}_{\Lambda}(-\vec{\rho}) \geq \Phi^{\gamma}_{\Lambda}(-\vec{\mu}) > 0\,. \tag{5.67}$$

Thus $\vec{\rho} \in \mathcal{R}_{\mathcal{P}}$, too, and $\mathcal{R}_{\mathcal{P}}$ is a down-set.

Remark. The proof is purely inductive as those in section 5.3.

Proof. As $\vec{\mu} \in \mathcal{R}_{\mathcal{P}}$ (5.10) holds for it. We show (5.67) by simultaneous induction over the cardinality of the Λ in (Λ, γ). Thus (5.10) holds for $\vec{\rho}$, too. The induction base is $\Lambda = \{\gamma\}$ and

$$0 < \Phi^{\gamma}_{\{\gamma\}}(-\vec{\mu}) = \Xi_{\{\gamma\}}(-\vec{\mu}) = 1 - \mu_\gamma \leq 1 - \rho_\gamma = \Xi_{\{\gamma\}}(-\vec{\rho}) = \Phi^{\gamma}_{\{\gamma\}}(-\vec{\rho}) \leq 1 \,.$$

For the induction step set $\{\xi_1, \dots, \xi_m\} := \mathcal{I}^\star(\gamma) \cap \Lambda$:

$$\begin{aligned}
&0 \\
&< \Phi^{\gamma}_{\Lambda}(-\vec{\mu}) && \text{as } \vec{\mu} \in \mathcal{R}_{\mathcal{P}} \\
&= 1 - \frac{\mu_\gamma}{\prod_{i=1}^m \Phi^{\xi_i}_{\Lambda \setminus \{\gamma, \xi_1, \dots, \xi_{i-1}\}}(-\vec{\mu})} && \text{by (5.3c)} \\
&\leq 1 - \frac{\mu_\gamma}{\prod_{i=1}^m \Phi^{\xi_i}_{\Lambda \setminus \{\gamma, \xi_1, \dots, \xi_{i-1}\}}(-\vec{\rho})} && \text{induction hypothesis} \\
&\leq 1 - \frac{\rho_\gamma}{\prod_{i=1}^m \Phi^{\xi_i}_{\Lambda \setminus \{\gamma, \xi_1, \dots, \xi_{i-1}\}}(-\vec{\rho})} && \text{as } \vec{\rho} \leq \vec{\mu} \\
&= \Phi^{\gamma}_{\Lambda}(-\vec{\rho}) && \text{by (5.3c)} \\
&\leq 1 && \text{by (5.3c).}
\end{aligned}$$

□

Proposition 164. *We have $\mathcal{R}_{\mathcal{P}} \subseteq [0,1[^{\mathcal{P}}$.*

Proof. Let $\vec{\rho} \in \mathcal{R}_{\mathcal{P}}$. Then for every $\gamma \in \mathcal{P}$ we have $\Xi_{\{\gamma\}}(-\vec{\rho}) = 1 - \rho_\gamma > 0$ and $\rho_\gamma < 1$. Thus $\vec{\rho} < \vec{1}$. □

If we take a different convex combination for each parameter we have more basic relations. For $\vec{\lambda} \in [0,1]^{\mathcal{P}}$ we have

$$\begin{aligned}
&- \log \Xi_\Lambda(-\vec{\lambda}\vec{\rho} - (\vec{1} - \vec{\lambda})\vec{\mu}) \\
&= \sum_{\vec{n}: \Lambda \to \mathbb{N}_0} a_{\vec{n}, \Lambda} \prod_{\xi \in \Lambda} (\lambda_\xi \rho_\xi + (1 - \lambda_\xi)\mu_\xi)^{n_\xi} \\
&\geq \sum_{\vec{n}: \Lambda \to \mathbb{N}_0} a_{\vec{n}, \Lambda} \prod_{\xi \in \Lambda} (\rho_\xi^{\lambda_\xi} \mu_\xi^{1-\lambda_\xi})^{n_\xi} \\
&= - \log \Xi_\Lambda(-\vec{\rho}^{\vec{\lambda}} \vec{\mu}^{\vec{1} - \vec{\lambda}})
\end{aligned}$$

and thus

$$\Xi_\Lambda(-\vec{\lambda}\vec{\rho} - (\vec{1} - \vec{\lambda})\vec{\mu}) \leq \Xi_\Lambda(-\vec{\rho}^{\vec{\lambda}} \vec{\mu}^{\vec{1} - \vec{\lambda}}) \,.$$

The function $\log \Xi_\Lambda$ is *sub-linear* for negative real fugacities

$$\begin{aligned}
&- \log \Xi_\Lambda(-\vec{\lambda}\vec{\rho} - (\vec{1} - \vec{\lambda})\vec{\mu}) \\
&= \sum_{n \geq 1} \frac{1}{n!} \sum_{\vec{\xi} \in \Lambda^n} \prod_{i=1}^n (\lambda_{\xi_i} \rho_{\xi_i} + (1 - \lambda_{\xi_i})\mu_{\xi_i}) \\
&\geq \sum_{n \geq 1} \frac{1}{n!} \sum_{\vec{\xi} \in \Lambda^n} \left(\prod_{i=1}^n \lambda_{\xi_i} \rho_{\xi_i} + \prod_{i=1}^n (1 - \lambda_{\xi_i})\mu_{\xi_i} \right) \\
&= - \log \Xi_\Lambda(-\vec{\lambda}\vec{\rho}) - \log \Xi_\Lambda(-(\vec{1} - \vec{\lambda})\vec{\mu})
\end{aligned}$$

and thus

$$\Xi_\Lambda(-\vec{\lambda}\vec{\rho} - (\vec{1} - \vec{\lambda})\vec{\mu}) \leq \Xi_\Lambda(-\vec{\lambda}\vec{\rho}) \, \Xi_\Lambda(-(\vec{1} - \vec{\lambda})\vec{\mu}) \,.$$

5.8.6 Proofs of analyticity

Proof of proposition 127. Let $\rho < \lambda_c$. We know that $F_{\mathcal{P}}$ is analytic in $-\vec{\rho}$ and thus finite in $[-\rho-\varepsilon, -\rho+\varepsilon]$, for some small ε. Thus

$$\begin{aligned}
&F_{\mathcal{P}}(-\rho-\varepsilon) < \infty \\
\Rightarrow\ &\forall\, \Lambda \Subset \mathcal{P}: \quad -\frac{1}{|\Lambda|}\log \Xi_\Lambda(-\rho-\varepsilon) < \infty \\
\Leftrightarrow\ &\forall\, \Lambda \Subset \mathcal{P}: \quad \Xi_\Lambda(-\rho-\varepsilon) > 0 \\
\Leftrightarrow\ &(\rho+\varepsilon)\vec{1} \in \mathcal{R}_{\mathcal{P}} \\
\Rightarrow\ &\rho\vec{1} \in \operatorname{Int}\mathcal{R}_{\mathcal{P}}\,.
\end{aligned}$$

Conversely, let $\rho \in [0,1]$ and $\varepsilon > 0$ with $(\rho+\varepsilon)\vec{1} \in \operatorname{Int}\mathcal{R}_{\mathcal{P}}$. Then

$$\begin{aligned}
&F_\Lambda(-\rho) \\
&= -\frac{1}{|\Lambda|}\log \Xi_\Lambda(-\rho) \\
&= -\frac{1}{|\Lambda|}\log \prod_{i=1}^{|\Lambda|} \Phi_{\Lambda_i}^{\xi_i}(-\rho) && \text{by the telescoping identity (5.3a)} \\
&\leq -\frac{1}{|\Lambda|}\log \prod_{\xi\in\Lambda} \frac{\varepsilon}{\rho+\varepsilon} && \text{by the generic bound(5.19)} \\
&= \frac{1}{|\Lambda|}\sum_{\xi\in\Lambda} -\log\frac{\varepsilon}{\rho+\varepsilon} \\
&= \log\frac{\rho+\varepsilon}{\varepsilon} \\
&< \infty
\end{aligned}$$

Thus $\rho < \lambda_c$. □

Proof of proposition 128. Fix $\Lambda \Subset \mathcal{P}$ and $\vec{\rho} \in \operatorname{Int}\mathcal{R}_{\mathcal{P}}$. Then there exists a $\vec{x} > 0$, such that $\vec{\rho} + \vec{x} \in \mathcal{R}_{\mathcal{P}}$. We write

$$\begin{aligned}
&F_\Lambda(-\vec{\rho}) \\
&= -\frac{1}{|\Lambda|}\log \Xi_\Lambda(-\vec{\rho}) \\
&= -\frac{1}{|\Lambda|}\log \prod_{i=1}^{|\Lambda|} \Phi_{\Lambda_i}^{\xi_i}(-\vec{\rho}) && \text{by the telescoping identity (5.3a)} \\
&\leq -\frac{1}{|\Lambda|}\log \prod_{\xi\in\Lambda} \frac{x_\xi}{\rho_\xi + x_\xi} && \text{by the generic bound(5.19)} \\
&< \infty
\end{aligned}$$

The expansion of $F_\Lambda(-\vec{\rho})$ as a series is given by (5.6a). □

Chapter 6

Pruned SAW tree interpretation and rigorous bounds on grid-like graphs

This sections aims to give some rigorous bounds for grid-like graphs, that is mainly Cayley graphs of $\mathbb{Z}^d$. These graphs are of particular interest to the mathematical physics community. The knowledge about them is diminished, though: unlike the graphs in section 2.6 the recursive calculation of the OVOEPs via the fundamental identity are not well-founded recursions. On the other hand their geometry allows *transfer-matrix* techniques [Sim93, II.5/II.6] [Bax89], which are outside the scope of this thesis, and results build on translation invariance.

Towards this aim we recall the pruned SAW (self-avoiding walk) tree interpretation by Scott & Sokal in section 6.2. We use this interpretation to deduce monotonicity under adding edges in section 6.3.1 and necessary conditions for $p \geq p_{sh}^G$, that is lower bounds on p_{sh}^G, for the above Cayley graphs in section 6.3.2.

The rest of this chapter deals with various aspects of $\mu_{\mathbb{Z}^d,\vec{p}}$: *tightness* of a certain class of OVOEPs in section 6.4.4, formalization of the *torus argument* for approximations in section 6.4.2 and *non-representation* as an *edge-factor* for p close to p_{sh}^G in section 6.4.3.

Exact solutions for $\mathbb{Z}^2$ have been obtained via *transfer-matrix techniques* and *series expansion techniques*. We list them and all rigorous bounds we know of, in section 6.5. This includes higher-depth bounds from chapter 5.

6.1 Reminder on notation

This chapter uses notation from both Shearer's measure in chapter 2 and polymer systems in chapter 5.

We recall some notation for trees. Let $\mathbb{T} := (V, E)$ be a tree with root o. For $v \in V$, we denote by $\mathbb{T}^v$ the *subtree induced by* v, that is induced by v and all its offspring. This effectively creating a new rooted tree rooted at v. The *level of a node* $v \in V$ is written $l(v)$ and $L(\mathbb{T}, n)$ is the n^{th} level of the tree. For $n \in \mathbb{N}$, we denote by $\mathbb{T}|_n$ the *level* n *restricted subtree of* $\mathbb{T}$, which is the subtree induced by v and all its offspring nodes with $l(v) \le n$.

6.2 A review of the pruned SAW tree interpretation by Scott & Sokal

This section reviews the pruned SAW tree interpretation by Scott & Sokal in the language of Shearer's measure. The original is in the language of statistical mechanics and distributed all over [SS05].

6.2.1 Exact calculation on finite trees

Algorithm 165 ([SS05, Algorithm T in section 3.4])**.** Let $\mathbb{T} := (V, E)$ be a finite tree rooted at o and $\vec{p} \in [0,1]^V$. Define the *effective parameters* by recursion upwards from the leaves towards the root:

$$q_v^{\mathrm{eff}} := q_v \prod_{w \in \mathfrak{C}(v) \colon q_w^{\mathrm{eff}} \neq 1} \frac{1}{1 - q_w^{\mathrm{eff}}} \,. \tag{6.1a}$$

In particular, if v is a leaf, then $q_v^{\mathrm{eff}} = q_v$. If $\vec{q}^{\mathrm{eff}}$ is well-defined, then we have the identity

$$\Xi_{\mathbb{T}}(\vec{p}) = \prod_{v \in V} (1 - q_v^{\mathrm{eff}}) \,. \tag{6.1b}$$

Remark. The vector $\vec{q}^{\mathrm{eff}}$ is called effective because it lets us treat $\mathbb{T}$ like a totally disconnected graph.

Proof of algorithm 165 by Temmel for $\vec{q} \in \mathring{\mathcal{Q}}_G^{sh}$. The fundamental identity (2.23) and factorization over disjoint subtrees (2.12) yield

$$\Xi_{\mathbb{T}}(\vec{p}) = \prod_{v \in V} \alpha^v_{V(\mathbb{T}^v) \setminus \{v\}}(\vec{p}) \,.$$

Hence (6.1b) follows by identifying q_v^{eff} with $1 - \alpha^v_{V(\mathbb{T}^v) \setminus \{v\}}(\vec{p})$ and remarking that (6.1a) is equivalent to the fundamental identity (2.23). □

Proof of algorithm 165 by Scott & Sokal. The proof uses by induction over the cardinality of V. If $V = \{v\}$, then $\mathfrak{C}(v) = \emptyset$. Hence $q_v^{\mathrm{eff}} = q_v$ and $1 - q_v^{\mathrm{eff}} = 1 - q_v = \Xi_{\mathbb{T}}(\vec{p})$. For the induction step suppose, that $\forall\, w \in \mathfrak{C}(o) : \Xi_{\mathbb{T}^w}(\vec{p}) = \prod_{v \in V(\mathbb{T}^w)} (1 - q_v^{\mathrm{eff}})$. Then

$$\begin{aligned}
&\prod_{v \in V} (1 - q_v^{\mathrm{eff}}) \\
&= (1 - q_o^{\mathrm{eff}}) \prod_{w \in \mathfrak{C}(o)} \prod_{v \in V(\mathbb{T}^w)} (1 - q_v^{\mathrm{eff}}) \\
&= \left(1 - q_o \prod_{w \in \mathfrak{C}(o) \colon q_w^{\mathrm{eff}} \neq 1} \frac{1}{1 - q_w^{\mathrm{eff}}} \right) \prod_{w \in \mathfrak{C}(o)} \prod_{v \in V(\mathbb{T}^w)} (1 - q_v^{\mathrm{eff}})
\end{aligned}$$

$$
\begin{aligned}
&= \left(\prod_{w\in\mathfrak{C}(o)} \prod_{v\in V(\mathbb{T}^w)} (1-q_v^{\mathrm{eff}}) \right) - q_o \left(\prod_{w\in\mathfrak{C}(o)} \prod_{v\in V(\mathbb{T}^w)\setminus\{w\}} (1-q_v^{\mathrm{eff}}) \right) \\
&= \prod_{w\in\mathfrak{C}(o)} \Xi_{\mathbb{T}^w}(\vec{p}) - q_o \prod_{w\in\mathfrak{C}(o)} \prod_{u\in\mathfrak{C}(w)} \prod_{v\in V(\mathbb{T}^u)} (1-q_v^{\mathrm{eff}}) \\
&= \prod_{w\in\mathfrak{C}(o)} \Xi_{\mathbb{T}^w}(\vec{p}) - q_o \prod_{w\in\mathfrak{C}(o)} \prod_{u\in\mathfrak{C}(w)} \Xi_{\mathbb{T}^w}(\vec{p}) \\
&= \Xi_{G(V\setminus\{o\})}(\vec{p}) - q_o \Xi_{G(V\setminus\mathcal{N}_1(o))}(\vec{p}) \\
&= \Xi_{\mathbb{T}}(\vec{p}).
\end{aligned}
$$

□

Proposition 166 ([SS05, theorem 3.2])**.** *Let $\mathbb{T}$ be a tree and $\vec{q}^{\,\mathit{eff}}$ as in (6.1a). Then*

$$\vec{q} \in \mathring{\mathcal{Q}}_G^{sh} \Leftrightarrow \vec{0} \le \vec{q}^{\,\mathit{eff}} < \vec{1}\,. \tag{6.2}$$

Proof by Temmel. We identify q_v^{eff} with $1-\alpha^v_{V(\mathbb{T}^v)\setminus\{v\}}(\vec{p})$ and remark that (6.1a) is equivalent to the fundamental identity (2.23). Thus (6.2) follows from (2.24). □

6.2.2 SAW trees

Let $G=(V,E)$ be a graph. A *path* or *walk* $P := \overline{v_0 \dots\ v_n}$ is a finite sequence of nodes such that $v_i \backsim v_{i+1}$. We call n the *length* of the walk P. If we also allow $v_i = v_{i+1}$ then P is *lazy*, while if we demand $v_i \neq v_j$ for all $i,j \in [n]_0$ then P is *self-avoiding*. Let $\mathrm{W}(G)$ be the set of all *finite length walks/paths* on the graph G. Let $\mathrm{SAW}(G)$ be the set of all *finite length self-avoiding walks/paths* on G and let $\mathrm{SAW}(G,v)$ be the set of all finite length self-avoiding walks/paths on G starting at the vertex v. We define the *prolongement operator* by

$$\rightsquigarrow:\quad \mathrm{W}(G)\times V \to \mathrm{W}(G) \quad \overline{v_1\dots v_n}\rightsquigarrow w := \begin{cases} P\rightsquigarrow w := \overline{v_1\dots v_n w} & \text{if } v_n \backsim w \\ P & \text{else.}\end{cases} \tag{6.3}$$

The *self-avoiding walk tree* $\mathbb{T}_{\mathrm{SAW}(G,o)} := (V',E')$ of G rooted at o has nodes $V' := \mathrm{SAW}(G,o)$. A path $P \in V'$ is the parent of a path Q iff there exists a $w \in V$ such that $P\rightsquigarrow w = Q$. Hence the root of $\mathbb{T}_{\mathrm{SAW}(G,o)}$ is the path $\overline{o}$ and $L(\mathbb{T}_{\mathrm{SAW}(G,o)}, n)$, the n^{th} level of $\mathbb{T}_{\mathrm{SAW}(G,o)}$, consists of all the *self-avoiding paths* of length n in G starting at o.

Definition 167 ([SS05, section 6.2, page 68])**.** For every self-avoiding path $P := \overline{ov_1\dots v_n} \in \mathrm{SAW}(G,o)$ choose a total order $\prec_P$ of its children in $\mathbb{T}_{\mathrm{SAW}(G,o)}$. The set of *spurned vertices* $\mathcal{S}(P)$ of P is then the set of smaller possible extensions of the parent of P:

$$\mathcal{S}(P) := \begin{cases} \emptyset & \text{if } P = \overline{o} \\ \{w \in V : Q\rightsquigarrow w \prec_Q Q\rightsquigarrow v\} & \text{if } P = Q\rightsquigarrow v\,. \end{cases} \tag{6.4}$$

Define the *pruned self-avoiding walk tree* $\mathbb{T}_{\mathrm{pSAW}(G,o)}$ of G based on o by pruning $P := \overline{ov_1\dots ov_n} \in \mathrm{SAW}(G,o)$ and all its children from $\mathbb{T}_{\mathrm{SAW}(G,o)}$ iff $\exists\, 1 \le i <$

$j \leq n$ with $v_j \in \mathcal{S}(\overline{ov_1 \ldots v_j})$. In other words, $P \in V(\mathbb{T}_{\mathrm{pSAW}(G,o)})$ iff

$$\forall 1 \leq i < j \leq n : \quad v_j \notin \mathcal{S}(\overline{o \ldots v_i}) \uplus \{v_i\} \,. \tag{6.5}$$

The graph G is a tree iff $G = \mathbb{T}_{\mathrm{SAW}(G,o)} = \mathbb{T}_{\mathrm{pSAW}(G,o)}$.

6.2.3 Tree interpretation

The following result originates in [SS05, section 6.2]. It is the result an unfolding procedure motivated by the recursive nature of the fundamental identity (2.11), which creates the pruned SAW tree.

Proposition 168 ([SS05, (6.9)])**.** *Let $G := (V,E)$ be a finite, connected graph. Choose a root $o \in V$. Fix $\mathbb{T}_{\mathrm{pSAW}(G,o)} := (V', E')$ by fixing the ordering among children in* SAW(G,o)*. For $\vec{p} \in [0,1]^V$, define the vector $\vec{\hat{p}} \in [0,1]^{V'}$ by*

$$\forall \overline{o \ldots v} \in V' : \quad \hat{q}_{\overline{o \ldots v}} := q_v \,. \tag{6.6a}$$

Define the accepted pair $(\overline{o \ldots v}, v)$ associated *with the path $\overline{o \ldots v}$ by*

$$\forall \overline{o \ldots v} \in V' : \quad \mathcal{A}(\overline{o \ldots uv}) := V \setminus \biguplus_{w \in \overline{o \ldots u}} \{w\} \setminus \bigcup_{w \in \overline{o \ldots v}} \mathcal{S}(\overline{o \ldots w}) \,. \tag{6.6b}$$

That is $\mathcal{A}(\overline{o \ldots v})$ are all vertices not visited or spurned prior to reaching v. In particular $v \in \mathcal{A}(\overline{o \ldots v})$ and $\mathcal{A}(\overline{o}) = V$. If $\vec{q} \in \mathring{\mathcal{Q}}^{sh}_G$, then

$$\forall \overline{o \ldots v} \in V' : \quad \hat{q}^{\mathit{eff}}_{\overline{o \ldots v}} = 1 - \alpha^v_{\mathcal{A}(\overline{o \ldots v})}(\vec{p}) \,. \tag{6.6c}$$

Proof. We prove (6.6c) by induction over the cardinality of V', upwards from the leaves. If $\overline{o \ldots v}$ is a leaf in $\mathbb{T}_{\mathrm{pSAW}(G,o)}$, i.e. a maximum pruned SAW, then v is isolated in $G(\Lambda(\overline{o \ldots v}))$. Hence $\alpha^v_{\Lambda(\overline{o \ldots v})}(\vec{p}) = p_v$. By algorithm T 165 we have

$$\hat{q}^{\mathrm{eff}}_{\overline{o \ldots v}} = \hat{q}_{\overline{o \ldots v}} = q_v = 1 - p_v = 1 - \alpha^v_{\mathcal{A}(\overline{o \ldots v})}(\vec{p}) \,.$$

For the induction step let $P := \overline{o \ldots v} \in V'$ be a non–leaf node. Suppose that (6.6c) holds for all extensions $P \rightsquigarrow w \in V'$. They are all children of P in $\mathbb{T}_{\mathrm{pSAW}(G,o)}$, hence we enumerate those w as $\{w_1, \ldots, w_m\}$ such that $w_m \prec_P w_{m-1} \prec_P \ldots w_2 \prec_P w_1$. Let $P_i := P \rightsquigarrow w_i$. Then algorithm T 165 yields

$$\begin{aligned}
&\hat{q}^{\mathrm{eff}}_P \\
&= \hat{q}_P \prod_{i=1}^m \frac{1}{1 - \hat{q}^{\mathrm{eff}}_{P_i}} && \text{by (6.1a)} \\
&= \hat{q}_P \prod_{i=1}^m \frac{1}{1 - (1 - \alpha^{w_i}_{\mathcal{A}(P_i)}(\vec{p}))} && \text{induction step} \\
&= q_v \prod_{i=1}^m \frac{1}{\alpha^{w_i}_{\mathcal{A}(P_i)}(\vec{p})} && \text{by (6.6a)} \\
&= q_v \prod_{i=1}^m \frac{1}{\alpha^{w_i}_{\mathcal{A}(P) \setminus \{v, w_i, \ldots, w_m\}}(\vec{p})} && \text{as } \mathcal{A}(P_i) = \mathcal{A}(P) \setminus \{v, w_i, \ldots, w_m\} \\
&= 1 - \alpha^v_{\mathcal{A}(P)}(\vec{p}) && \text{by the fundamental identity (2.23)} \,.
\end{aligned}$$

□

Theorem 169 ([SS05, (6.12)]). *For all choices of $o \in V$ and orderings among children as required in definition 167 we have*

$$\vec{p} \in \mathring{\mathcal{P}}_{sh}^{G} \Leftrightarrow \vec{p} \in \mathring{\mathcal{P}}_{sh}^{\mathbb{T}_{\mathrm{pSAW}(G,o)}}\,, \tag{6.7}$$

where $\vec{\tilde{p}}$ is defined as in (6.6a).

Proof. Fix o and $\mathbb{T}_{\mathrm{pSAW}(G,o)} = (V', E')$ by fixing the ordering among children in V' as required in definition 167. Equation (6.2) tells us that $\vec{\tilde{q}} \in \mathring{\mathcal{Q}}^{sh}_{\mathbb{T}_{\mathrm{pSAW}(G,o)}}$ iff $\vec{0} \leq \vec{\tilde{q}}^{\,\mathrm{eff}} < \vec{1}$. By (6.6c) this amounts to:

$$\forall\, P = \overline{o \ldots v} \in V' : \quad 0 \leq \hat{q}_P^{\mathrm{eff}} = 1 - \alpha^{v}_{\mathcal{A}(P)}(\vec{p}) < 1 \quad \Leftrightarrow \quad \alpha^{v}_{\mathcal{A}(P)}(\vec{p}) \in]0,1]\,. \tag{6.8}$$

If $\vec{q} \in \mathring{\mathcal{Q}}^{sh}_{G}$, then $\alpha^{v}_{W}(\vec{p}) > 0$ for all (W, v) and the rhs of (6.8) holds. Hence $\vec{\tilde{q}} \in \mathring{\mathcal{Q}}^{sh}_{\mathbb{T}_{\mathrm{pSAW}(G,o)}}$. On the other hand, for every (W, v) there exists a pruned SAW tree and node $P_{W,v}$ in that tree such that $\Lambda(P_{(W,v)}) = W \uplus \{v\}$. If $\vec{\tilde{q}} \in \mathring{\mathcal{Q}}^{sh}_{\mathbb{T}_{\mathrm{pSAW}(G,o)}}$, then $\alpha^{v}_{W}(\vec{p}) \in]0,1]$. Hence $\forall\, W \subseteq V$, W independent, we have $\Xi_{G(W)}(\vec{p}) > 0$, which by (2.14) is equivalent to $\vec{q} \in \mathring{\mathcal{Q}}^{sh}_{G}$. □

6.2.4 Inequality of arithmetic and geometric means

This section recalls some analytic facts needed in section 6.2.5.

Lemma 170 (Generalized AM-GM inequality [Heu80, 59.1]). *Let $\vec{x} \geq \vec{0}$ and $\vec{a} > \vec{0}$ be two vectors of length n. Let $a := \sum_{i=1}^{n} a_i$, then*

$$\frac{1}{a} \sum_{i=1}^{n} a_i x_i \geq \left(\prod_{i=1}^{n} x_i^{a_i} \right)^{1/a}\,, \tag{6.9}$$

with equality iff $\vec{x} = x\vec{1}$, for some $x \geq 0$.

Proposition 171. *Let $\alpha, \beta > 0$ and $f_{\alpha,\beta} : z \mapsto \frac{\beta}{(1-z)^{\alpha}}$. If $1 > x \geq y \geq 0$, then $f_{\alpha,\beta}(x) \geq f_{\alpha,\beta}(y)$.*

Proof. $\frac{\partial f_{\alpha,\beta}}{\partial z}(z) = \frac{\alpha\beta}{(1-z)^{\alpha+1}} > 0$ and $\frac{\partial^2 f_{\alpha,\beta}}{\partial z^2}(z) = \frac{\alpha(\alpha+1)\beta}{(1-z)^{\alpha+2}} > 0$ for $z \neq 1$. Hence $f_{\alpha,\beta}$ is strongly convex. □

6.2.5 Tree homogenization bounds

The following propositions show first how to homogenize effective parameters of a tree at one level in proposition 172 and then how to extend this to the average of a finite nuber of levels in proposition 173. It all boils down to four applications of the generalized inequality of arithmetic and geometric means in lemma 170 to bound means of $\vec{q}^{\,\mathrm{eff}}$.

Proposition 172 ([SS05, Proposition 3.3]). *Let $l_i := |L(\mathbb{T}, i)|$ and $g_i := l_{i+1}/l_i$. For $\vec{q} \in [0,1]^V$ define $(\bar{q}_i)_{i=0}^{D}$ by $\bar{q}_i := \left(\prod_{v \in L(\mathbb{T},i)} q_v \right)^{1/l_i}$. Define $(\hat{q}_i)_{i=0}^{D}$ recursively by $\hat{q}_D := \bar{q}_D$ and (while $\hat{q}_{i+1} < 1$)*

$$\hat{q}_i := \frac{\bar{q}_i}{(1 - \hat{q}_{i+1})^{g_i}}\,. \tag{6.10}$$

If there is a level i such that $\hat{q}_i \geq 1$, then $\vec{q} \notin \mathring{\mathcal{Q}}^{sh}_{\mathbb{T}}$.

Proof. Let $\vec{q}^{\text{eff}}$ be as in (6.1a). Define $(\bar{q}_i^{\text{eff}})_{i=0}^D$ by

$$\bar{q}_i^{\text{eff}} := \left(\prod_{v\in L(\mathbb{T},i)} q_v^{\text{eff}}\right)^{1/l_i} = \left(\prod_{v\in L(\mathbb{T},i)} \frac{q_v}{\prod_{w\in\mathfrak{C}(v)}(1-q_w^{\text{eff}})}\right)^{1/l_i}$$
$$= \frac{\bar{q}_i}{\left(\prod_{v\in L(\mathbb{T},i+1)}(1-q_v^{\text{eff}})\right)^{1/l_i}} .$$

Two applications of the arithmetic-geometric mean inequality (6.9) result in

$$\left(\prod_{v\in L(\mathbb{T},i+1)}(1-q_v^{\text{eff}})\right)^{1/l_{i+1}} \leq 1 - \frac{1}{l_{i+1}}\sum_{v\in L(\mathbb{T},i+1)} q_v^{\text{eff}}$$
$$\leq 1 - \left(\prod_{v\in L(\mathbb{T},i+1)} q_v^{\text{eff}}\right)^{1/l_{i+1}} = 1 - \bar{q}_{i+1}^{\text{eff}} ,$$

which in turn implies that

$$\bar{q}_i^{\text{eff}} \geq \frac{\bar{q}_i}{(1-\bar{q}_{i+1}^{\text{eff}})^{g_i}} . \tag{6.11}$$

This has the form $f_{g_i,\bar{q}_i}$ from proposition 171. Note that $\bar{q}_D^{\text{eff}} = \bar{q}_D = \hat{q}_D$. Apply the inequality from proposition 171 recursively to see that $(\bar{q}_i^{\text{eff}})_{i=0}^D \geq (\hat{q}_i)_{i=0}^D$. If there is a level i such that $\bar{q}_i^{\text{eff}} \geq \hat{q}_i \geq 1$, then proposition 166 asserts that $\vec{q} \notin \mathring{\mathcal{Q}}^{sh}_{\mathbb{T}}$. □

Proposition 173 ([SS05, Proposition 3.4])**.** *We extend the statement of proposition 172. Fix $2 \leq k \leq D$ and choose $G \in [1,\infty[$. Let $\gamma_i := l_i/G^i$ and $y_{i,k} := \sum_{j=i}^{i+k-1}\gamma_j$. For $\vec{q} \in [0,1]^V$ define $(\bar{q}_{i,k})_{i=1}^{D+1-k}$ by $\bar{q}_{i,k} := \left(\prod_{j=1}^{i+k-1}\bar{q}_j^{\gamma_j}\right)^{1/\gamma_{i,k}}$. Define $(\hat{q}_{i,k})_{i=0}^{D+1-k}$ recursively by $\hat{q}_{D+1-k,k} := \bar{q}_{D+1-k,k}$ and (while $\hat{q}_{i+1,k} < 1$)*

$$\hat{q}_{i,k} := \frac{\bar{q}_{i,k}}{(1-\hat{q}_{i+1,k})^{G\gamma_{i+1,k}/\gamma_{i,k}}} . \tag{6.12}$$

If there is a level i such that $\hat{q}_{i,k} \geq 1$, then $\vec{q} \notin \mathring{\mathcal{Q}}^{sh}_{\mathbb{T}}$.

Proof. Let $\vec{q}^{\text{eff}}$ be as in (6.1a). Define $(\bar{q}_{i,k}^{\text{eff}})_{i=0}^{D+1-k}$ by

$$\bar{q}_{i,k}^{\text{eff}}$$
$$:= \left(\prod_{j=i}^{i+k-1}(\bar{q}_j^{\text{eff}})^{\gamma_j}\right)^{1/\gamma_{i,k}}$$
$$\geq \left(\prod_{j=i}^{i+k-1}\left(\frac{\bar{q}_j}{(1-\bar{q}_{j+1}^{\text{eff}})^{g_j}}\right)^{\gamma_j}\right)^{1/\gamma_{i,k}} \qquad \text{by (6.11)}$$

$$= \frac{\left(\prod_{j=i}^{i+k-1} \bar{q}_j^{\gamma_j}\right)^{1/\gamma_{i,k}}}{\left(\prod_{j=i}^{i+k-1}(1-\bar{q}_{j+1}^{\text{eff}})^{g_j\gamma_j}\right)^{1/\gamma_{i,k}}}$$

$$= \frac{\bar{q}_{i,k}}{\prod_{j=i}^{i+k-1}(1-\bar{q}_{j+1}^{\text{eff}})^{G\gamma_{j+1}/\gamma_{i,k}}} \qquad \text{as } g_j\gamma_j = \frac{l_{j+1}}{l_j}\frac{l_j}{G^j} = B\frac{l_{j+1}}{G^{j+1}} = G\gamma_{j+1}\,.$$

Two applications of the arithmetic-geometric mean inequality (6.9) result in

$$\prod_{j=i}^{i+k-1}(1-\bar{q}_{j+1}^{\text{eff}})^{\gamma_{j+1}/\gamma_{i+1,k}} \leq 1 - \sum_{j=i}^{i+k-1}\frac{\gamma_{j+1}}{\gamma_{i+1,k}}\bar{q}_{j+1}^{\text{eff}}$$
$$\leq 1 - \prod_{j=1}^{i+k-1}\left(\bar{q}_{j+1}^{\text{eff}}\right)^{\gamma_{j+1}/\gamma_{i+1,k}} = 1 - \bar{q}_{i+1,k}^{\text{eff}}\,,$$

which in turn implies that

$$\bar{q}_{i,k}^{\text{eff}} \geq \frac{\bar{q}_{i,k}}{(1-\bar{q}_{i+1,k}^{\text{eff}})^{G\gamma_{i+1,k}/\gamma_{i,k}}}\,.$$

This has the form $f_{\alpha,\beta}$ with $\alpha = G\gamma_{i+1,k}/\gamma_{i,k}$ and $\beta = \bar{q}_{i,k}$ from proposition 171. Note that $\bar{q}_{D+1-k,k}^{\text{eff}} = \bar{q}_{D+1-k,k} = \hat{q}_{D+1-k,k}$. Apply the inequality from proposition 171 recursively to see that $(\bar{q}_{i,k}^{\text{eff}})_{i=0}^{D+1-k} \geq (\hat{q}_{i,k})_{i=0}^{D+1-k}$. If there is a level i such that $\bar{q}_{i,k}^{\text{eff}} \geq \hat{q}_{i,k} \geq 1$, then there is a level $j \in \{i,\ldots,i+k-1\}$ with $\bar{q}_j^{\text{eff}} \geq 1$ and proposition 166 asserts that $\vec{q} \notin \mathring{\mathcal{Q}}_{\mathbb{T}}^{sh}$. □

6.2.6 Growth number bounds for homogenized trees

This section extends section 6.2.5 to the infinite case. Define the *upper growth rate* of a tree $\mathbb{T}$ rooted at o by

$$\overline{\text{gr}}(\mathbb{T}) := \limsup_{n\to\infty} |L(\mathbb{T},n)|^{1/n}\,. \tag{6.13}$$

The infinite necessary condition on $\vec{q}$ is

Theorem 174 ([SS05, Proposition 8.3])**.** *Let $\mathbb{T} := (V,E)$ be an infinite tree rooted at $o \in V$. Let $G := \overline{\text{gr}}(\mathbb{T})$ and $\forall\, i \in \mathbb{N} : l_i := |L(\mathbb{T},i)|$. Every vector $\vec{q} \in [0,1]^V$ satisfying*

$$\exists\, Q > \frac{G^G}{(G+1)^{(G+1)}} : \forall\, i \in \mathbb{N} : \quad \bar{q}_i = \left(\prod_{v\in L(\mathbb{T},i)} q_v\right)^{1/l_i} \geq Q \tag{6.14}$$

is not in $\mathring{\mathcal{Q}}_G^{sh}$.

Note to self: Because you seem to take too long to get the inequalities right: $B := \text{br}(\mathbb{T}) \leq \overline{\text{gr}}(\mathbb{T}) =: G$, hence $\frac{B^B}{(B+1)^{(B+1)}} \geq \frac{G^G}{(G+1)^{(G+1)}}$ and the branching number $\text{br}(\mathbb{T})$ can never play a role.
◊

We start with two technical propositions:

Proposition 175 ([SS05, Lemma 8.5]). *Let $D \geq 1$, let $g_0, \dots, g_{D-1} > 0$ and define $G = \left(\prod_{i=0}^{D-1} g_i\right)^{1/D}$. Let $Q \geq 0$. Suppose there exist $q_0, \dots, q_D \in [0,1[$ satisfying*

$$\forall\, i \in \{0, \dots, D-1\} : \quad q_i \geq \frac{Q}{(1-q_{i+1})^{g_i}} \tag{6.15a}$$

and

$$q_0 \leq q_D. \tag{6.15b}$$

Then $Q \leq \frac{G^G}{(G+1)^{(G+1)}}$.

Proof. When $D = 1$ then (6.15) yields

$$Q \leq q_0(1-q_1)^{g_0} \leq q_1(1-q_1)^{g_0}\,,$$

hence $Q \leq \frac{G^G}{(G+1)^{(G+1)}}$ as $G = g_0$ and by applying proposition 52.

If $D > 1$ then define weights $y_i = \left(\prod_{j=0}^{i-1} g_j\right)/G^i$ for $i \in \{0, \dots, D\}$ and let $\Gamma = \sum_{i=0}^{D-1} \gamma_i$. Note that $\gamma_0 = \gamma_D = 1$. Define

$$\hat{q}_0 = \left(\prod_{i=0}^{D-1} q_i^{\gamma_i}\right)^{1/\Gamma} \quad \text{and} \quad \hat{q}_1 = \left(\prod_{i=1}^{D} q_i^{\gamma_i}\right)^{1/\Gamma}.$$

The discussion in the proof of proposition 173 implies that

$$\hat{q}_0 \geq \frac{Q}{(1-\hat{q}_1)^G}$$

On the other hand, as $q_0 \leq q_D$ and $\gamma_0 = \gamma_D = 1$ we have

$$\hat{q}_0 = \left(q_0 \prod_{i=1}^{D-1} q_i^{\gamma_i}\right)^{1/\Gamma} \leq \left(q_D \prod_{i=1}^{D-1} q_i^{\gamma_i}\right)^{1/\Gamma} = \hat{q}_1$$

Conclude by applying the argument for $D = 1$ to $\hat{q}_0$ and $\hat{q}_1$. □

Proposition 176 ([SS05, Lemma 8.4]). *Let $(g_i)_{i=0}^{\infty}$ be a sequence of positive real numbers, let $G = \limsup_{k\to\infty}(g_0 \cdots g_{D-1})^{1/D}$ and let $Q \geq 0$. Suppose there exists a sequence $(q_i)_{i=0}^{\infty}$ satisfying*

$$\forall\, i : \quad 0 \leq q_i < 1 \quad \textit{and} \quad q_i \geq \frac{Q}{(1-q_{i+1})^{g_i}}, \tag{6.16}$$

then $Q \leq \frac{G^G}{(G+1)^{(G+1)}}$

Proof. If $Q > \frac{G^G}{(G+1)^{(G+1)}}$ then choose for $\varepsilon > 0$ a $H < G$ such that

$$\tilde{Q} := \frac{Q}{1+\varepsilon} > \frac{H^H}{(H+1)^{(H+1)}} > \frac{G^G}{(G+1)^{(G+1)}}. \tag{6.17}$$

This is possible by (2.51c). Since $G > H$ we can find indices $(i_m)_{m\in\mathbb{N}}$ satisfying $1 \le i_1 < i_2 < i_3 < \cdots$ and

$$\forall\, m : \prod_{i=i_m+1}^{i_{m+1}} g_i \ge H^{i_{m+1}-i_m}. \tag{6.18}$$

As $Q > 0$ and by (6.16) we see that $\forall\, i : q_i \ge \frac{Q}{(1-q_{i+1})^{g_i}} > Q > 0$. Now, if $\forall\, m \in \mathbb{N} : q_{i_m} > (1+\varepsilon)q_{i_{m+1}}$, then

$$0 < Q \le \liminf_{m\to\infty} q_{i_m} \le \liminf_{m\to\infty} \frac{q_{i_1}}{(1+\varepsilon)^m} = 0,$$

a contradiction. Hence there must be an index M such that $q_{i_M} \le (1+\varepsilon)q_{i_{M+1}}$. For ease of notation we now let $D = i_{M+1}-i_M$, let $\hat{q}_j = q_{i_M}+i$ for $j \in \{0,\dots,D\}$ and let $\hat{b}_j = g_{i_M}+j$ for $j \in \{0,\dots,D-1\}$. Now (6.16) translates into

$$\forall\, j \in \{0,\dots,D-1\} : \quad \hat{q}_j \ge \frac{Q}{(1-\hat{q}_{j+1})^{\hat{b}_j}}.$$

Furthermore $\hat{q_0} \le (1+\varepsilon)\hat{q_D}$. If we set $\tilde{q}_j = \hat{q}_j$ for $j \in \{0,\dots,D\}$ and $\tilde{q}_0 = \hat{q}_0/(1+\varepsilon)$, then

$$\forall\, j \in \{0,\dots,D-1\} : \quad \tilde{q}_j \ge \frac{\tilde{Q}}{(1-\tilde{q}_{j+1})^{\hat{b}_j}}.$$

By proposition 175 and (6.18) we get that $\tilde{Q} \le \frac{H^H}{(H+1)^{(H+1)}}$, while (6.17) asserts that $\tilde{Q} > \frac{H^H}{(H+1)^{(H+1)}}$. Hence a contradiction and $Q \le \frac{G^G}{(G+1)^{(G+1)}}$. □

Proof. Proof of theorem 174 Suppose $\vec{q} \in \mathring{\mathcal{Q}}^{sh}_{\mathbb{T}}$ and fulfils (6.14). Then

$$\forall\, D \in \mathbb{N} : \vec{q} \in \mathcal{Q}^{sh}_{\mathbb{T}|_D}.$$

Define $g_i = l_{i+1}/l_i$ and $(\hat{q}_i^{(D)})_{i=0}^{\infty}$ by

$$\hat{q}_i^{(D)} = \begin{cases} \frac{\bar{q}_i}{\left(1-\hat{q}_{i+1}^{(D)}\right)^{g_i}} & \text{if } i \in \{0,\dots,D-1\} \\ \bar{q}_D & \text{if } i = D \\ 0 & \text{else.} \end{cases}$$

It follows from the proof of proposition 172 that on $\mathbb{T}|_D$

$$\forall\, i \in \{0,\dots,D\} : \quad 0 \le \hat{q}_i^{(D)} \le \bar{q}_i^{\text{eff}} < 1\,.$$

Furthermore, by the monocity of $f_{\alpha,\beta}$ in proposition 171, we have

$$\forall\, i \in \mathbb{N} : \qquad \hat{q}_i := \limsup_{i\to\infty} \hat{q}_i^{(D)} \le 1,$$

where $(\hat{q}_i)_{i=0}^{\infty}$ fulfils the inequality

$$\forall\, i \in \mathbb{N} : \qquad \hat{q}_i = \frac{\bar{q}_i}{(1-\hat{q}_{i+1})^{g_i}} \ge \frac{Q}{(1-\hat{q}_{i+1})^{g_i}}.$$

From the inequality we can deduce that $(1-\hat{q}_i)^{g_i} \ge Q > \frac{G^G}{(G+1)^{(G+1)}} > 0$, hence $\hat{q}_i \le \left(1 - \frac{G^G}{(G+1)^{(G+1)}}\right)^{1/g_i} < 1$ for all $i \in \mathbb{N}$. The statement now follows from proposition 176. □

6.3 Building on the pruned SAW tree interpretation

This section shows some applications of the pruned SAW tree interpretation from section 6.2.3.

6.3.1 Monotonicity under adding edges

We first extend theorem 169 to the infinite case. Then we use it to show monotonicity of the various quantities connected to Shearer's measure under adding edges, something that we have not been able to do in chapter 2. In the same vein we treat covers of graphs in proposition 179.

Proposition 177. *For every graph $G := (V, E)$, all choices of $o \in V$ and orderings among children as required in definition 167, we have*

$$\vec{p} \in \mathring{\mathcal{P}}_{sh}^{G} \Leftrightarrow \vec{\hat{p}} \in \mathring{\mathcal{P}}_{sh}^{\mathbb{T}_{\mathrm{pSAW}(G,o)}}, \tag{6.19}$$

where $\vec{\hat{p}}$ is defined as in (6.6a).

Proof. This is a direct consequence of theorem 169 and the definition of $\mathring{\mathcal{P}}_{sh}^{G}$ in (2.34b). ☐

Proposition 178. *Let $G := (V, E)$ and $G' := (V, E')$ be two graphs sharing on the same set of vertices. The set of admissible parameters and the OVOEPs are monotone in the edges. Formally*

$$E \subseteq E' \Rightarrow \mathcal{P}_{sh}^{G} \supseteq \mathcal{P}_{sh}^{G'}. \tag{6.20a}$$

Writing $(\alpha_{\cdot}^{\cdot})'$ for the OVOEPs with respect to G', we have

$$\forall (W, v), \vec{p} \in \mathcal{P}_{sh}^{G'}: \quad \alpha_W^v(\vec{p}) \geq (\alpha_W^v)'(\vec{p}). \tag{6.20b}$$

Remark. This proposition adresses the points made after propositions 21, 32 and 26. See also lemma 117 for a proof via cluster expansion.

Proof. We prove proposition (6.20a) by induction over the cardinality of $E' \setminus E$. Thus we regard the case $E' \setminus E =: \{e =: (v, w)\}$. We construct both $\mathbb{T}_{\mathrm{pSAW}(G,v)}$ and $\mathbb{T}_{\mathrm{pSAW}(G',v)}$ using the same local orders, extending all $\prec_P$ with P ending in v to include v and vice-versa. Therefore $\mathbb{T}_{\mathrm{pSAW}(G,v)} \leq \mathbb{T}_{\mathrm{pSAW}(G',v)}$. A bigger tree implies more constraints on $\vec{q}$, thus $\mathcal{P}_{sh}^{G} \supseteq \mathcal{P}_{sh}^{G'}$. This proves (6.20a).

To show (6.20b) again add just one edge $e := (u, w)$. The crucial case is $v = u$ and $w \in W$. With $\{w_1, \dots, w_m\} := W \cap \mathcal{N}(v)$ in G and $\{w, w_1, \dots, w_m\} := W \cap \mathcal{N}(v)$ in G' we have

$$\begin{aligned}
&\alpha_W^v(\vec{p}) \\
&= 1 - \frac{q_v}{\prod_{i=1}^{m} \alpha_{W \setminus \{v, w_{i+1}, \dots, w_m\}}^{w_i}(\vec{p})} \\
&\leq 1 - \frac{q_v}{\alpha_{W \setminus \{v, w_1, \dots, w_m\}}^{w} \prod_{i=1}^{m} \alpha_{W \setminus \{v, w_{i+1}, \dots, w_m\}}^{w_i}(\vec{p})}
\end{aligned}$$

$$= (\alpha^v_W)'(\vec{p}) .$$

If $\{u, w\} \not\subseteq W \uplus \{v\}$, then (W, v) is unaffected by the presence of e. If $\{u, w\} \subseteq W$, then use simultaneous induction over all v with respect to the cardinality of W over all such (W, v). □

Proposition 179. *Let G be a* cover *(in the topological sense of 1-complexes) of a connected graph H, then $p^H_{sh} \leq p^G_{sh}$.*

Remark. The lifting is inspired by the comment in [SS05, example 8.4 on page 87].

Proof. Fix $o \in V(H)$ and let $\tilde{o} \in V(G)$ be a lift of o. For every vertex in $\mathbb{T}_{\mathrm{pSAW}(H,o)}$ lift the path it represents into $\mathbb{T}_{\mathrm{pSAW}(G,\tilde{o})}$. We choose as the total ordering among children in $\mathbb{T}_{\mathrm{pSAW}(G,\tilde{o})}$ an extension of the total ordering in $\mathbb{T}_{\mathrm{pSAW}(H,o)}$. Hence $\mathbb{T}_{\mathrm{pSAW}(H,o)} \leq \mathbb{T}_{\mathrm{pSAW}(G,\tilde{o})}$. Use (6.19) and (6.20a) to get

$$p^H_{sh} = p^{\mathbb{T}_{\mathrm{pSAW}(H,o)}}_{sh} \leq p^{\mathbb{T}_{\mathrm{pSAW}(G,\tilde{o})}}_{sh} = p^G_{sh} .$$

□

6.3.2 A necessary condition for admissibility

The necessary condition on $\vec{q}$ in the case of trees in theorem 174 translates into the following bound for arbitrary graphs

Theorem 180. *Let $G := (V, E)$ be an infinite graph. Let $g := \overline{\mathrm{gr}}(\mathbb{T}_{\mathrm{pSAW}(G,o)})$, for some root vertex $o \in V$ and choice of orders as in definition 167. Then*

$$p^G_{sh} \geq 1 - \frac{g^g}{(g+1)^{(g+1)}} . \tag{6.21}$$

Proof. We use the bijection from (6.19) along the homogeneous cross-section and see that $p^G_{sh} = p^{\mathbb{T}_{\mathrm{pSAW}(G,o)}}_{sh}$. Conclude by applying the upper growth number bound (6.14). □

6.3.3 An inductive partition scheme dual to Scott & Sokal's pruned SAW tree interpretation

This section is dedicated to a modification of *Ret*, such as to become dual to the tree-interpetation of the optimal bound for nonnegativity of Ξ_Λ, reviewed in section 6.2.3.

Recall that the tree-interpretation uses an unfolding of the partition function via the fundamental identity, choosing for each application of the fundamental identity an arbitrary total order of $\mathcal{I}^\star(\gamma)$. Then the rhs of the fundamental identity is developed according to the ordering.

The same concept, applied to *Ret* and the cluster level would be to use this total order (like in the tree-interpretation chosen arbitrarily for each path in the history) every time we branch into a non-same component: only retain those vertices as candidates, which are the tallest children for their potential parents in the boundary. This implies that all the smaller siblings of the chosen parent become forbidden, too, mirroring the pruned SAW tree interpretation on the polymer system level.

Let I be a finite, totally ordered set. Given $\vec{\xi} \in \mathcal{P}^I$ we let $G := G(\vec{\xi})$ and assume that it is connected, that is it is a cluster. We present an inductive partition scheme *Dual* adapted to clusters. The static information comprises the total order on I, the structure of G as cluster of $\vec{\xi}$, the choice of root $o \in I$ and, for every $P \in \mathrm{SAW}(\mathcal{P})$ ending in γ, a total order $<_P$ on $\mathcal{I}^\star(\gamma)$.

Algorithm 181 (*Dual* exploration)**.** Let $H \in \mathcal{C}_G$. For every k let H_k, T_k, U_k, B_k and P_k be as in algorithm 135. The missing parts to construct H_{k+1} from H_k are:

To $i \in T_k$ we *associate the polymer walk* given by

$$w : \quad I \to \mathrm{SAW}(\mathcal{P}) \quad w(i) := (\xi_j)_{j \in P(o,i)} \,. \tag{6.22}$$

The polymer walk $w(i)$ may be lazy. The *non-lazy version* $\tilde{w}(i)$ of $w(i)$ removes repetitions of the same polymer. Define the *accepted set* $\mathcal{A}(i)$ of i as

$$\mathcal{A}(i) := \mathcal{A}(\tilde{w}(i)) \,. \tag{6.23}$$

Call an edge $(i,j) \in E(C \cap P_k, B_k) \cap E(H)$ *accepted* (or **A**) iff $\xi_i \in \mathcal{A}(j)$ and *rejected* (or **R**) iff $\xi_i \not\in\in \mathcal{A}(j)$. Likewise call a vertex $i \in P_k$ rejected iff all such (i,j) are rejected and accepted iff there exists such an accepted (i,j). Finally we say that a connected component C of $H_k|_{U_k}$ is rejected iff all vertices in $C \cap P_k$ are rejected and accepted iff $C \cap P_k$ contains at least one accepted vertex.

If C is an **R** connected component of $H_k|_{U_k}$:

(dba) SELECT $C \cap S_k := C \cap P_k$.

(dia) As $E(B_k, (C \cap P_k) \setminus (C \cap S_k)) = \emptyset$. Thus REMOVE nothing.

(dpa) For $i \in C \cap S_k$, let $j_i := \mathrm{argmin}\,\{j \in B_k : (i,j) \in E(H_k)\}$ and SELECT (i, j_i).

(dua) For $i \in C \cap S_k$, REMOVE all $(i,j) \in E(H_k)$ with $j_i \neq j \in B_k$.

(dca) REMOVE all of $E(C \cap S_k) \cap E(H_k)$.

If C is a **A** connected component of $H_k|_{U_k}$:

(dbr) SELECT $C \cap S_k := \{i \in C \cap P_k : i \text{ is } \mathbf{R}\}$.

(dir) REMOVE all of $E(B_k, (C \cap P_k) \setminus (C \cap S_k)) \cap E(H_k)$.

(dpr) For $i \in C \cap S_k$, let $j_i := \mathrm{argmin}\,\{j \in B_k : (i,j) \in E(H_k) \text{ is } \mathbf{A}\}$ and SELECT (i, j_i).

(durr) For $i \in C \cap S_k$, REMOVE every **A** $(i,j) \in E(H_k)$ with $j_i \neq j \in B_k$.

(dura) For $i \in C \cap S_k$, REMOVE every **R** $(i,j) \in E(C \cap S_k, B_k) \cap E(H_k)$.

(dcr) REMOVE all of $E(C \cap S_k) \cap E(H_k)$.

The construction mirrors the one in the exploration algorithm 151 for *Ret*. Replace *same* **S** and *different* **D** with *rejected* **R** and *accepted* **A** respectively.

Algorithm 182 (*Dual* tree edge complement partition)**.** Let $\mathbb{T} \in \mathcal{T}_G$. Let L_k be the k^{th} level of $\mathbb{T}$. We have again a status function

$$s: \quad I \setminus \{o\} \to \{\mathbf{A}, \mathbf{R}\} \quad i \to \begin{cases} \mathbf{A} & \text{if } i \in \mathcal{A}(\mathfrak{p}(i)) \\ \mathbf{R} & \text{if } i \notin \mathcal{A}(\mathfrak{p}(i))\,. \end{cases} \tag{6.24}$$

Do the same construction for equivalence relations and equivalence classes on each level of $\mathbb{T}$ as in algorithm 153. This yields a tree structure on equivalence classes.

Let $0 \leq k \leq l$, $j \in L_k$, $i \in L_l$ and $e := (i,j) \in E \setminus E(\mathbb{T})$. Then $e \in \mathcal{A}_{Dual}(\mathbb{T})$ iff one of the mutually exclusive conditions (6.25) holds:

$$[j]_{(k)} = [i]_{(l)}\,, \tag{6.25a}$$

$$l \geq 2 \wedge [j]_{(k)} \in P([o]_{(0)}, [\mathfrak{p}(\mathfrak{p}(i))]_{(l-2)}) \wedge \xi_i \notin \mathcal{A}(j) \wedge s(C) = \mathbf{A}\,, \tag{6.25b}$$

where $C \in P([j]_{(k)}, [\mathfrak{p}(i)]_{(l-1)})$ the unique class with $\mathfrak{p}(C) = [j]_{(k)}$,

$$l \geq 1 \wedge [j]_{(k)} = \mathfrak{p}([i]_{(l)}) \wedge \xi_i \in \mathcal{A}(j) \wedge s(i) = \mathbf{A} \wedge j > \mathfrak{p}(i)\,, \tag{6.25c}$$

$$l \geq 1 \wedge [j]_{(k)} = \mathfrak{p}([i]_{(l)}) \wedge \xi_i \notin \mathcal{A}(j) \wedge s(i) = \mathbf{A}\,, \tag{6.25d}$$

$$l \geq 1 \wedge [j]_{(k)} = \mathfrak{p}([i]_{(l)}) \wedge \xi_i \notin \mathcal{A}(j) \wedge s(i) = \mathbf{R} \wedge j > \mathfrak{p}(i)\,. \tag{6.25e}$$

Remark. Of particular importance are the *deep edges* in (6.25b). This is where we use the particular structure of $G(\vec{\xi})$. Paths in the tree returning to a previously visited or spurned polymer find here always an admissible edge to add, thus excluding the tree from $\mathcal{T}_{Dual}(G(\vec{\xi}))$.

Proposition 183. *The map*

$$Dual: \quad \mathcal{T}_G \to \mathcal{C}_G \quad \mathbb{T} \mapsto Dual(\mathbb{T}) := (I, E(\mathbb{T}) \uplus \mathcal{A}_{Dual}(\mathbb{T})) \tag{6.26}$$

is a partition scheme of G with $[\mathbb{T}, Dual(\mathbb{T})] = \mathcal{E}^{-1}_{Dual}(\mathbb{T})$.

Proof. The proof is the same as the proof of proposition 154. We have again two clases and the is given after the dual exploration algorithm 181. □

Proposition 184 (Properties of $\mathcal{T}_{Dual}(G(\vec{\xi}))$)**.** *Let $\mathbb{T} \in \mathcal{T}_{Dual}(G(\vec{\xi}))$ and let C_i be the set of children of i in $\mathbb{T}$. Then*

$$|C_i| = |\operatorname{supp} \vec{\xi}_{C_i}| \tag{6.27a}$$

$$(\operatorname{supp} \vec{\xi}_{C_i}) \setminus \{\xi_i\} \textit{ is an independent subset of } \mathcal{I}^\star(\xi_i) \tag{6.27b}$$

$$\forall\, k \in \mathbb{N}_0, i \in L_k: \quad \operatorname{supp} \vec{\xi}_{[i]_{(k)}} \textit{ is an independent subset of } \mathcal{P} \tag{6.27c}$$

$$\forall\, i \in I \setminus \{o\}: \quad i \textit{ is } \mathbf{R} \Rightarrow \xi_i \notin \operatorname{supp} \vec{\xi}_{P(o,\mathfrak{p}(i))} \setminus \{\xi_i\} \tag{6.27d}$$

$$\forall\, i \in I \setminus \{o\}: \quad i \textit{ is } \mathbf{R} \Rightarrow \xi_i \notin \bigcup_{j \in P(o,i) \setminus \{o\}} \{\varepsilon \in \mathcal{I}^\star(\xi_{\mathfrak{p}(j)}) : \varepsilon <_{\tilde{w}(\mathfrak{p}(j))} \xi_j\}\,. \tag{6.27e}$$

Proof. This proof is the same as the proof of proposition 155, with the same modification as in the proof of proposition 183. The lazy self-avoiding (6.27d) and pruned (6.27e) properties of the polymer level walk from the root are a consequence of the deep edges (6.25b). □

Comparing the pruned SAW tree property (6.5) with (6.27e) and (6.27d), we see that $\xi_{P(o,i)}$ is a lazy, pruned self-avoiding path with respect to the same family of orders indexed by $\mathrm{SAW}(\mathcal{P})$.

6.4 Cayley graphs of $\mathbb{Z}^d$

This section focuses on a Cayley graph G of $\mathbb{Z}^d$, finitely generated by an asymmetric set $S := \{\vec{s_1}, \dots, \vec{s_{\kappa_G}}\}$ of cardinality κ_G. This implies that

$$\forall\, \vec{x} \in \mathbb{Z}^d : \quad \mathcal{N}(\vec{x}) = \{\vec{x} \pm \vec{s} : s \in S\} \quad \wedge \quad |\mathcal{N}(\vec{x})| = 2\kappa_G\,.$$

The graph G is translation-invariant under the natural action (translations) of $\mathbb{Z}^d$.

6.4.1 Subtrees of pruned SAW trees

In this section we determine subtrees of pruned SAW trees amenable to analysis. This yields *upper bounds on* $q_{sh}^{\cdot}$ for the pruned SAW trees containing them and subsequently for G. We introduce *translation invariant separation orders* and use them to construct three such examples of subtrees: the vectorized pruned SAW tree in section 6.4.1, the *dimensional pruned SAW tree* in section 6.4.1 and the *stacked pruned SAW tree* in section 6.4.1. This leads to general upper bounds in theorem 187 and the precise asymptotic behaviour for $\mathbb{Z}^d$ in theorem 192.

Proposition 185. *Let* $\mathbb{T}$ *be an infinite subtree of* $\mathbb{T}_{\mathrm{pSAW}(G,\vec{0})}$ *with respect to some orders as in definition 167. If* $\mathfrak{g} := \mathrm{gr}(\mathbb{T})$*, then*

$$q_{sh}^{G} \le \frac{\mathfrak{g}^{\mathfrak{g}}}{(\mathfrak{g}+1)^{(\mathfrak{g}+1)}}\,. \tag{6.28}$$

Proof. Theorem 180 asserts that

$$q_{sh}^{G} = q_{sh}^{\mathbb{T}_{\mathrm{pSAW}(G,\vec{0})}} \le q_{sh}^{\mathbb{T}} = \frac{\mathfrak{g}^{\mathfrak{g}}}{(\mathfrak{g}+1)^{(\mathfrak{g}+1)}}\,.$$

□

Definition 186. We call a vector $\vec{0} < \vec{a} \in \mathbb{R}^d$ with $\mathbb{Q}$-linear independent entries a *separation vector* of dimension d. It defines a *total separation order* $\prec_{\vec{a}}$ on $\mathbb{Z}^d$ compatible with addition

$$\vec{x} \prec_{\vec{a}} \vec{y} \Leftrightarrow \langle \vec{a}, \vec{x} \rangle < \langle \vec{a}, \vec{y} \rangle\,. \tag{6.29}$$

This order divides the neighbourhood of a vertex into two hemispheres:

$$\begin{aligned} \forall\, \vec{x} \in \mathbb{Z}^d : \quad \mathcal{N}(\vec{x})^{+(-)} &:= \{\vec{x} + \vec{s} : \vec{s} \in S \uplus S^{-1}, \langle \vec{a}, \vec{s} \rangle > (<)\, 0\} \\ &= \{\vec{y} \in \mathcal{N}(\vec{x}) : \langle \vec{a}, \vec{y} \rangle < (>) \langle \vec{a}, \vec{x} \rangle\}\,. \end{aligned} \tag{6.30}$$

Let $P := \overline{\vec{x_0} \dots \vec{x}_n} \in \mathrm{W}(G)$ and $\vec{s} \in S \uplus S^{-1}$. The *extension* of P by a step in direction $\vec{s}$ is written as

$$P \overset{+}{\rightsquigarrow} \vec{s} := \overline{\vec{0} \dots \vec{x}_n (\vec{x}_n + \vec{y})} = P \rightsquigarrow (\vec{x}_n + \vec{s}) \,.$$

We combine rooted trees with the *binary, non-commutative, left-associative graft operation* $\mathbb{T}_1 \odot \mathbb{T}_2$, grafting $\mathbb{T}_2$ onto $\mathbb{T}_1$ by regarding $\mathbb{T}_1$'s root as a child $\mathbb{T}_2$'s root.

If a tree $\mathbb{T}$ is defined by a finite recursive equation (ignoring depth), then the *growth rate* $\mathrm{gr}(\mathbb{T})$ exits [LP11, section 3.3] and is equal to its upper growth rate:

$$\mathrm{gr}(\mathbb{T}) := \lim_{n\to\infty} |L(\mathbb{T}, n)|^{1/n} = \overline{\mathrm{gr}}(\mathbb{T}) \,.$$

Vectorized pruned SAW tree

A first estimate. Inspired by and generalizing [SS05, example 8.3, page 89]. It has the disadvantage of not being asymptotically tight, which is achieved in section 6.4.1).

Theorem 187.

$$p^G_{sh} \geq 1 - \frac{{\kappa_G}^{\kappa_G}}{(\kappa_G + 1)^{(\kappa_G+1)}} \,. \tag{6.31}$$

Definition 188. Let $\vec{a}$ be a separation vector. Define the *vectorized, pruned SAW-tree* $\mathbb{T}^{\vec{a}}_{\mathrm{vSAW}}(P)$ of G rooted at $P \in \mathrm{SAW}(G)$ recursively by

$$\mathbb{T}^{\vec{a}}_{\mathrm{vSAW}}(P) := G(\{P\}) \odot \bigodot_{\vec{y} \in \mathcal{N}(\vec{0})^+} \mathbb{T}^{\vec{a}}_{\mathrm{vSAW}}(P \overset{+}{\rightsquigarrow} \vec{y}) \,. \tag{6.32}$$

Proposition 189. *The tree* $\mathbb{T}^{\vec{a}}_{\mathrm{vSAW}}(\overline{\vec{0}})$ *is a subtree of some* $\mathbb{T}_{\mathrm{pSAW}(\mathbb{Z}^d, \vec{0})}$.

Proof. For $Q := \overline{x_0 \dots x_n} \in \mathrm{W}(G)$, define $Q_j := \overline{x_0 \dots x_j}$ and $a(Q) := \langle \vec{a}, \vec{x_n} \rangle$. Define the positive self-avoiding walks by

$$\mathrm{SAW}(G, x_0)^+ := \{\overline{x_0 \dots x_n} \in \mathrm{W}(G) : \forall\, i \in [n] : x_n - x_{n-1} \in \mathcal{N}(\vec{0})^+\} \,.$$

They are clearly self-avoiding as $a(Q_j)$ is strictly increasing in j for every $Q \in \mathrm{SAW}(G)^+$. It follows from (6.32) that

$$V(\mathbb{T}^{\vec{a}}_{\mathrm{vSAW}}(\overline{\vec{0}})) = \mathrm{SAW}(G, \vec{0})^+ \,. \tag{6.33}$$

For $P \in \mathrm{SAW}(G, \vec{0})^+$ and $\vec{x}, \vec{y} \in \mathcal{N}(\vec{0})^+$ define a total order $\prec_P$ by

$$P \overset{+}{\rightsquigarrow} \vec{x} \prec_P P \overset{+}{\rightsquigarrow} \vec{y} \quad \Leftrightarrow \quad \vec{x} \prec_{\vec{a}} \vec{y} \,. \tag{6.34}$$

By the identification (6.33) $\prec_{\vec{a}}$ is a total order on the children of P in $\mathbb{T}^{\vec{a}}_{\mathrm{vSAW}}(\overline{\vec{0}})$. Let $P := \overline{\vec{0}\vec{x}_1 \dots \vec{x}_n} \in \mathrm{SAW}(G)^+$ with $n \geq 1$. Le

$$Q \in \mathrm{SAW}(G, \vec{0})^+ \setminus \mathcal{A}(P) = \bigcup_{j=0}^{n} (\{P_j\} \cup \mathcal{S}(P_j)) \,.$$

We want to show, that for every $\vec{y} \in \mathcal{N}(\vec{0})^+$, $P \overset{+}{\rightsquigarrow} \vec{y} \neq Q$. If this holds, then $\mathbb{T}^{\vec{a}}_{\text{vSAW}}(\overline{\vec{0}})$ fulfils the pruned SAW tree property (6.5) with respect to $\prec_P$ from (6.34).

By the above choice of Q we have $j \in [n-1]_0$ and either $\vec{y} \in \mathcal{N}(\vec{0})^+$ with $\vec{y} \prec_{\vec{a}} \vec{x}_{j+1} - \vec{x}_j$ or $\vec{y} := \vec{0}$, such that $Q = P_j \overset{+}{\rightsquigarrow} \vec{y}$. It follows, that

$$a(Q) = a(P_j) + \langle \vec{a}, \vec{y} \rangle < a(P_{j+1}) \leq a(P)$$

and $P \neq Q$. □

Proof. Proof of theorem 187 The tree $\mathbb{T}^{\vec{a}}_{\text{vSAW}}(P)$ is a rooted, regular rank κ_G tree, therefore $\operatorname{gr}(\mathbb{T}^{\vec{a}}_{\text{vSAW}}(P)) = \kappa_G$. Conclude by proposition 185. □

Dimensional pruned SAW tree

The idea is that for m consecutive steps we always step in a new dimension, then forget everything except the dimension chosen in the first step and reiterate. This way we guarantee that at each step we have at least $2(d-m)$ choices, except in the forgetting step. This tree is a subtree of the much more general stacked, pruned SAW tree (6.41), of whom it keeps the essential properties.

Definition 190. Fix $0 < m \leq d$. Let $S := \{\vec{\mathbf{e}}_1, \ldots, \vec{\mathbf{e}}_d\}$ and $C := (c_1, \ldots, c_l)$ a stack of elements of $S \uplus S^{-1}$ pictured growing to the right. Define the *free coordinates* $F(C)$ of C by $F(C) := \{i \in [d] : \pm\vec{\mathbf{e}}_i \notin C\}$. Define the *dimensional, pruned SAW tree* $\mathbb{T}^{m,C}_{\text{dSAW}}(P)$ of depth m of $\mathbb{Z}^d$ rooted at $P \in \text{SAW}(G)$ and indexed by the stack C of *committed directions* recursively by

$$\mathbb{T}^{m,C}_{\text{dSAW}}(P) := \begin{cases} G(\{P\}) \odot \bigodot_{i \in I(C)} \mathbb{T}^{m, C \uplus \{\vec{\mathbf{e}}_i\}}_{\text{dSAW}}(P \overset{+}{\rightsquigarrow} \vec{\mathbf{e}}_i) & \\ \qquad \odot \bigodot_{i \in I(C)} \mathbb{T}^{m, C \uplus \{-\vec{\mathbf{e}}_i\}}_{\text{dSAW}}(P \overset{-}{\rightsquigarrow} \vec{\mathbf{e}}_i) & \text{if } |C| < m \\ G(\{P\}) \odot \mathbb{T}^{m, \{\vec{\mathbf{e}}_d\}}_{\text{dSAW}}(P \overset{+}{\rightsquigarrow} \vec{\mathbf{e}}_d) & \text{else.} \end{cases} \tag{6.35}$$

Proposition 191. *The tree $\mathbb{T}^{m,(\vec{\mathbf{e}}_d)}_{\text{dSAW}}(\overline{\vec{0}})$ is a subtree of some $\mathbb{T}_{\text{pSAW}(\mathbb{Z}^d, \vec{0})}$.*

Proof. It follows from (6.35) that every stack $C := (c_1, \ldots, c_l)$ contains l different vectors and $l \leq m$. Hence we can define the total order $\prec_C$ on $S \uplus S^{-1}$ by

$$-c_1 \prec_C \cdots \prec_C -c_l \prec_C \underbrace{\cdots\cdots\cdots}_{\{\pm\vec{\mathbf{e}}_i : i \in I(C)\} \text{ arbitrarily ordered}} \prec_C c_l \prec_C \cdots \prec_C c_1 \,. \tag{6.36}$$

Let $\overline{\vec{0}} \neq P := \overline{\vec{0}\vec{x}_1 \ldots \vec{x}_n} \in V(\mathbb{T}^{m,(\vec{\mathbf{e}}_d)}_{\text{dSAW}}(\overline{\vec{0}}))$ with $n \geq 1$ and let $C_P := (c_1, \ldots, c_{l_P})$ be the stack at the time P was reached in the recursion (6.35). There exists a $k_P \in \mathbb{N}_0$ such that the relation $n = k_P m + l_P$ holds. We also define the total order $\prec_P$ among all possible extensions (not only self-avoiding ones) of P by

$$P \overset{+}{\rightsquigarrow} \vec{x} \prec_P P \overset{+}{\rightsquigarrow} \vec{y} \Leftrightarrow \vec{x} \prec_{C_P} \vec{y} \,. \tag{6.37}$$

The walk P increases its d^{th} coordinate exactly every m^{th} step. Let $P_j := \overline{\vec{0}\ldots\vec{x}_j}$. Hence for every $0 \leq j \leq (k_P m \wedge n) - 1$ we have in the d^{th} coordinate

$$\vec{x}_j^d < \vec{x}_{k_P m}^d = \vec{x}_n^d\,.$$

For every C appearing in (6.35) we always have $\vec{\mathbf{e}}_d \in C$ and $-\vec{\mathbf{e}}_d \notin C$. Hence

$$\forall\,\vec{y} \in S \uplus S^{-1}: \quad \vec{y} \prec_{C_{P_j}} \vec{x}_{j+1} - \vec{x}_j \;\Rightarrow\; \vec{y} \neq \vec{\mathbf{e}}_d \;\Rightarrow\; \vec{x}_j^d + \vec{y}^d < \vec{x}_{k_P m}^d = \vec{x}_n^d\,,$$

hence $P \notin \mathcal{S}(P_j)$. In the case of $k_P m \leq j \leq n-1$ we see that our total order $\prec_{C_{P_j}}$ remembers the directions committed to and puts them always highest. Therefore the same reasoning as in the d^{th} coordinate applies:

$$\begin{aligned}\forall\,\vec{y} \in S \uplus S^{-1}: \;\; & \vec{y} \prec_{C_{P_j}} \vec{x}_{j+1} - \vec{x}_j =: \varepsilon_j \vec{\mathbf{e}}_{i_j} \\ & \Rightarrow \vec{y} \notin \{\vec{\mathbf{e}}_d\} \uplus \{\varepsilon_l \vec{\mathbf{e}}_{i_l} : k_P m \leq l \leq j\} \\ & \Rightarrow \forall\, i \notin F(C_{P_j}) : \vec{x}_j^i + \vec{y}^i < \vec{x}_n^i \vee \vec{x}_j^i + \vec{y}^i > \vec{x}_n^i\,,\end{aligned}$$

hence $P \notin \mathcal{S}(P_j)$. Thus $\mathbb{T}_{\text{vSAW}}^{\vec{a}}(\vec{\vec{0}})$ fulfils the pruned SAW tree property (6.5) with respect to $\prec_P$ from (6.37). □

Theorem 192.

$$p_{sh}^{\mathbb{Z}^d} \geq 1 - \frac{\mathfrak{g}^{\mathfrak{g}}}{(\mathfrak{g}+1)^{(\mathfrak{g}+1)}}, \tag{6.38}$$

where $\mathfrak{g} := \sqrt[m]{\prod_{l=1}^{m-1} 2(d-l)}$.

Proof. Rooted subtrees of $\mathbb{T}_{\text{dSAW}}^{m,\{\vec{\mathbf{e}}_d\}}(\vec{\vec{0}})$ with the same $|C|$ are isomorphic under relabelling of the coordinates and flipping of their orientation. Furthermore $\mathbb{T}_{\text{dSAW}}^{m,\{\vec{\mathbf{e}}_d\}}(\vec{\vec{0}})$ is periodic and we know [LP11, section 3.3] that

$$\operatorname{gr}(\mathbb{T}_{\text{dSAW}}^{m,\{\vec{\mathbf{e}}_d\}}(\vec{\vec{0}})) = \sqrt[m]{\prod_{l=1}^{m-1} 2(d-l)}\,. \tag{6.39}$$

Conclude by applying proposition 185. □

Corollary 193. *We have*

$$\lim_{d\to\infty} d\, q_{sh}^{\mathbb{Z}^d} = \frac{1}{2e}. \tag{6.40}$$

Remark. In the proof we choose $m = m(d)$ to balances two goals: on the one hand m should grow fast enough to compensate for the missing m^{th} term in (6.39), while on the other hand it should grow slow enough so that the factors in (6.39) don't become to small compared to $2d$.

Proof. Fix $0 < \alpha < 1$. Let $m : \mathbb{N} \to \mathbb{N} \quad d \mapsto m(d) \in [d]$ be such that $\lim_{d\to\infty} \frac{d^\alpha}{m(d)} = c > 0$. We minorize

$$\mathfrak{g}(d) = \sqrt[m(d)]{\prod_{l=1}^{m(d)-1} 2(d-l)} \geq [2(d-m(d))]^{1-\frac{1}{m(d)}}\,.$$

Using that $\limsup_{d\to\infty} \frac{m(d)}{d} = 0$ and $\lim_{d\to\infty} m(d) = \infty$ this yields:

$$\begin{aligned}
\liminf_{d\to\infty} \frac{\mathfrak{g}(d)}{2d} &\geq \liminf_{d\to\infty} \frac{[2(d-m(d)]^{1-\frac{1}{m(d)}}}{2d} \\
&\geq \liminf_{d\to\infty} \frac{2(d-m(d))}{2d} \liminf_{d\to\infty} \frac{1}{\sqrt[m(d)]{2(d-m(d))}} \\
&= \liminf_{d\to\infty} \left(1 - \frac{m(d)}{d}\right) \liminf_{m\to\infty} \frac{1}{\sqrt[m]{2(c\,m^{1/\alpha} - m)}} \\
&= \left(1 - \limsup_{d\to\infty} \frac{m(d)}{d}\right) \liminf_{m\to\infty} \frac{1}{(\sqrt[m]{m})^{1/\alpha}} \\
&= \left(\liminf_{m\to\infty} \frac{1}{\sqrt[m]{m}}\right)^{1/\alpha} = 1^{1/\alpha} = 1\,.
\end{aligned}$$

As $\mathfrak{g}(d) \leq 2d$ this implies that $\lim_{d\to\infty} \frac{\mathfrak{g}(d)}{2d} = 1$. Use (2.50) to get

$$\begin{aligned}
\lim_{d\to\infty} d \frac{\mathfrak{g}(d)^{\mathfrak{g}(d)}}{(\mathfrak{g}(d)+1)^{(\mathfrak{g}(d)+1)}} &= \lim_{d\to\infty} \frac{2d}{\mathfrak{g}(d)} \lim_{d\to\infty} \frac{\mathfrak{g}(d)}{2} \frac{\mathfrak{g}(d)^{\mathfrak{g}(d)}}{(\mathfrak{g}(d)+1)^{(\mathfrak{g}(d)+1)}} \\
&= \frac{1}{2} \lim_{g\to\infty} g \frac{g^g}{(g+1)^{(g+1)}} = \frac{1}{2e}\,.
\end{aligned}$$

We know that $\frac{\mathfrak{g}(d)^{\mathfrak{g}(d)}}{(\mathfrak{g}(d)+1)^{(\mathfrak{g}(d)+1)}} \geq q_{sh}^{\mathbb{Z}^d} \geq \frac{(2d-1)^{(2d-1)}}{(2d)^{(2d)}}$ and conclude by multiplying with d and taking limits as in (2.50). $\square$

Stacked pruned SAW tree

This is the original idea underlying the dimensional, pruned SAW tree. It is more general in the following ways:

- There is no cap on the size of the stack, that is it may grow to depth d.
- One can take more than one step in a committed direction.
- When forgetting committed directions one may only pop some elements off the stack, instead of all except the bottom-most one.

It is also more complicated to calculate with than the dimensional pruned SAW tree. We state it only for completeness.

Definition 194. Let $S := \{\vec{\mathbf{e}}_1, \dots, \vec{\mathbf{e}}_d\}$ and $C := (c_1, \dots, c_l)$ be a stack of elements of $S \uplus S^{-1}$. Define the *free dimensions* $F(C) := \{i \in [d] : \pm\vec{\mathbf{e}}_i \notin C\}$ of C. Define the *stacked, pruned SAW tree* $\mathbb{T}^C_{\mathrm{sSAW}}(P)$ of $\mathbb{Z}^d$ rooted at $P \in \mathrm{SAW}(G)$ and indexed by the stack C of *committed directions* recursively by

$$\begin{aligned}\mathbb{T}^{C}_{\mathrm{sSAW}}(P) := G(\{P\}) \\ &\odot \mathbb{T}^{(c_1)}_{\mathrm{sSAW}}(P \overset{+}{\rightsquigarrow} c_1) \\ &\odot \mathbb{T}^{(c_1,c_2)}_{\mathrm{sSAW}}(P \overset{+}{\rightsquigarrow} c_2) \\ &\odot \dots \\ &\odot \mathbb{T}^{C}_{\mathrm{sSAW}}(P \overset{+}{\rightsquigarrow} c_l) \\ &\odot \bigodot_{i \in F(C)} \mathbb{T}^{(c_1,\dots,c_l,\vec{\mathbf{e}}_i)}_{\mathrm{sSAW}}(P \overset{+}{\rightsquigarrow} \vec{\mathbf{e}}_i) \\ &\odot \bigodot_{i \in F(C)} \mathbb{T}^{(c_1,\dots,c_l,-\vec{\mathbf{e}}_i)}_{\mathrm{sSAW}}(P \overset{-}{\rightsquigarrow} \vec{\mathbf{e}}_i)\,.\end{aligned} \tag{6.41}$$

Proposition 195. *The periodic tree $\mathbb{T}^{(\vec{\mathbf{e}}_d)}_{\mathrm{sSAW}}(\vec{\vec{0}})$ is a subtree of some $\mathbb{T}_{\mathrm{pSAW}(\mathbb{Z}^d,\vec{0})}$.*

Proof. The same idea as in the proof of proposition 191, adapted to the less restrictive constraints. In short, it is again a dynamic drift argument with recursion over the dimension d: if $C := (c_1, \dots, c_l)$ then we have only $d - l$ free coordinates to choose from and the total orders make the l coordinates committed to irrelevant. □

The growth of $\mathbb{T}^{(\vec{\mathbf{e}}_d)}_{\mathrm{sSAW}}(\vec{\vec{0}})$ is encoded (again using isomorphism via relabelling of substrees with same stack depths) by the largest real eigenvalue of the lower $d \times d$ Hessenberg matrix $M(d)$ with entries

$$M(d)_{(i,j)} := \begin{cases} 1 & \text{if } j < i+1 \Leftrightarrow j \le i \\ 2(d-i) & \text{if } j = i+1 \\ 0 & \text{if } j \ge i+1\,. \end{cases} \tag{6.42}$$

Its characteristic polynomial $p_d(\lambda) = \det(\lambda I - M(d))$ fulfils

$$p_d(\lambda) = (\lambda - 1)p_{d-1}(\lambda) - \sum_{i=0}^{d-2} p_i(\lambda) \prod_{j=i+1}^{d-1} (2j) \tag{6.43a}$$

$$p_d(\lambda) = d\, p_{d-1}(\lambda) \tag{6.43b}$$

$$p_d(0) = -\prod_{i=1}^{d-1} (2i - 1). \tag{6.43c}$$

6.4.2 The torus argument

To justify taking the limit in the *transfer-matrix* approach by Todo [Tod99] and Guttman [Gut87] a torus argument is used. The proof of this argument itself is always missing, though. We give a rigorous proof of this argument, following the outline in [SS05, example 8.4 on page 87]. For background information see the connection with analyticity in section 5.2.3.

Proposition 196. *Let G be the cover of a graph H by the action of a subgroup Γ of $\mathbb{Z}^d$, then $p^H_{sh} \le p^G_{sh}$. If $\{\Gamma_n\}_{n\in\mathbb{N}}$ a sequence of subgroups of $\mathbb{Z}^d$ such that their fundamental domains are growing and exhaust $\mathbb{Z}^d$, then we have $\lim_{n\to\infty} p^{H_n}_{sh} = p^G_{sh}$.*

Proof. The first part is a rephrasing of the cover proposition 179. For the second part let $B_n := [-n, n]^d$ and $G_n := G(B_n)$ the subgraph of G on the at $\vec{0}$ centred cube of side-length $2n + 1$. As the fundamental domains exhaust $\mathbb{Z}^d$ we have

$$\forall\, n \in \mathbb{N} : \exists\, M = M(n) \in \mathbb{N} : \forall\, m \geq M : \quad G_n \leq H_m \;\Rightarrow\; q_{sh}^{H_m} \leq q_{sh}^{G_n} .$$

Putting it together with the first part we get

$$\lim_{m\to\infty} p_{sh}^{H_m} \leq p_{sh}^{G} = \lim_{n\to\infty} p_{sh}^{G_n} \leq \lim_{m\to\infty} p_{sh}^{H_m} .$$

□

6.4.3 Construction as an edge-factor

If we regard the intrinsic construction from section 2.6.1 in the particular case of $G := Z$ (with nearest-neighbour edges), then we get a particular kind of *zero-one switch* on each edge (see figure 2.1, but for $k = 1$). A natural idea is to put such a switch on each edge, thus inhibiting its endpoints to ever realize in 0 simultaneously. Model 200 shows, that this is possible, but proposition 201 only for high p and not for all of $[p_{sh}^{G}, 1]$.

Definition 197. Take a regular graph $G := (V, E)$ with degree D. Let $Y := \{Y_v\}_{v\in V}$ and $X := \{X_e\}_{e\in E}$ be processes, with X_e being identical $\hat{X}$-distributed. We call Y a *edge factor of* X *on* G iff there exists a measurable function f from $\hat{X}^D$ and bijective *index functions* $\{ind_v\}_{v\in V}$ with $ind_v : [D] \to \mathcal{N}(v)$, such that

$$\forall\, v \in V : \quad Y_v := f((X_{(v,ind_v(i))})_{i=1}^{D}) . \tag{6.44}$$

Definition 198. Let $G := (V, E)$ have uniformly bounded vertex degrees. An *orientation* O is a function $E \to V$, mapping each edge to its endpoint, yielding a direction for the edge. For an orientation O, we define the *in-and outdegrees* $deg_{in}, deg_{out} : V \mapsto \mathbb{N}_0$. We also have

$$\begin{aligned} D_{in}^{min} &:= \min\{deg_{in}(v) : v \in V\} & D_{in}^{max} &:= \max\{deg_{in}(v) : v \in V\} \\ D_{out}^{min} &:= \min\{deg_{out}(v) : v \in V\} & D_{out}^{max} &:= \max\{deg_{in}(v) : v \in V\} . \end{aligned} \tag{6.45}$$

If they coincide we write $D_{in}^{max} := D_{in}^{min} = D_{in}^{max}$, as well as $D_{out}^{max} := D_{out}^{min} = D_{out}^{max}$.

Proposition 199. *Let G a connected graph admitting an orientation with either $D_{in}^{max} = 1$ or $D_{out}^{max} = 1$. Then G is either a* horocyclic tree *[CKW94] or a finite size circle with trees branching out from that circle.*

Proof. Using symmetry we just consider the case $D_{in}^{max} = 1$. Draw the oriented graph given by the orientation, then along a directed path you either return to your node (possible for at most one circle because of connectedness) or branch out without ever coming back (the tree parts). □

Model 200. Take a graph G with maximum vertex degree D. For a given orientation O let $D' := D_{in}^{max} + D_{out}^{max}$. For $p \in [1 - \frac{D_{in}^{max} D_{in}^{max} D_{out}^{max} D_{out}^{max}}{D'^{D'}}, 1]$ let c be the bigger solution of $q = c^{D_{out}^{max}}(1-c)^{D_{in}^{max}} = h_{D_{out}^{max}, D_{in}^{max}}(c)$ (2.49). We construct $\mu_{G,p}$ as a factor of a BPF on the edges of an extended $G' := (V, E')$ with parameter c.

Proof. For $v \in V$ we have ${D_{in}^{max}}_v$ edges oriented towards v. Add $D_{in}^{max} - {D_{in}^{max}}_v$ *phantom edges* leading from v out to one common phantom stub node (which we do not care about). Proceed likewise for ${D_{out}^{max}}_v$ and edges oriented away from v. After having this done for all of V, create a BPF $\Pi_c^{E'}$, where c as above and $E' := E \uplus \{\text{phantom edges}\}$. Create a BRF $Y := (Y_v)_{v\in V}$ by

$$\forall\, v \in V : \quad 1 - Y_v := \Big(\prod_{e\in Out(v)} X_e\Big)\Big(\prod_{f\in In(v)} (1 - X_f)\Big)\,. \tag{6.46}$$

The independent sets in G and G' are the same. For an independent $W \Subset V$ we have $Y_W \sim \Pi_p^W$. Along each edge we have a *zero-one switch*: for $e = (v, w) \in E$ oriented $v \to w$ the term X_e appears in Y_v and $(1 - X_e)$ in Y_w, hence the products in (6.46) can never be simultaneously be 1. Thus Y_v and Y_w never realize in 0 simultaneously. The characterization (2.8) asserts that Y is $\mu_{G,p}$-distributed. □

Proposition 201. *Let $G := (V, E)$ be a D-regular graph and $Y := \{Y_v\}_{v\in V} \sim \mu_{G,p}$. Then for $p < 1 - \frac{(D-1)^{(D-1)}}{D^D}$ we cannot write Y as an edge factor of an iid field of Uniform($[0, 1]$)-distributed rvs $X := \{X_e\}_{e\in E}$.*

Remark. The non-realization result of proposition 201 takes inspiration from [AGKdV89, section 3], which treats 2-factors on $\mathbb{Z}$. It should not come as a big surprise - the construction in model 200 does not even take the structure of $\mathcal{N}(v)$ into account.

Proof. Suppose that Y is an edge factor of such an X. Then there exists $\{ind_v\}_{v\in V}$ as in definition 197 and $\emptyset \neq A \subsetneq [0, 1]^D$ *measurable* such that

$$\forall\, v \in V : \quad Y_v = \mathbb{I}_{A^c}((X_{(v,ind_v(i))})_{i=1}^D)\,.$$

For $i \in [D]$, define the *index collection* $I(i) \in \mathcal{P}([D])$ with

$$I(i) := \{j \in [D] : \exists\, v, w \in V : ind_v(j) = w\}\,,$$

the *marginal integral* $F_i : [0, 1] \to [0, 1]$ with

$$F_i(x) := \int \cdots \int \mathbb{I}_A(y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_D) dy_1 \cdots dy_{i-1} dy_{i+1} \cdots dy_D$$

and the *marginal support* A_i by

$$A_i := \{x \in [0, 1] : F_i(x) > 0\}\,.$$

If $v \backsim w$ with $ind_v(i) = w$ and $ind_w(j) = v$, then we get

$$0 = \mathbb{P}(Y_v = 0, Y_w = 0) = \int_{[0,1]} F_i(x)F_j(x)dx = \int_{A_i\cap A_j} F_i(x)F_j(x)dx\,.$$

Let λ be the Lebesgue measure on $[0, 1]$. We have

$$i \in I(j) \;\Rightarrow\; A_i \cap A_j = \emptyset \quad \lambda - a.s. \;\Rightarrow\; A_i \subseteq A_j^c \quad \lambda - a.s. \quad .$$

Hence $\forall\, i \in [D] : i \notin I(i)$, as $A_i \cap A_i^c = \emptyset$ λ-a.s. .

As it is obvious that $\forall\, i \in [D] : \emptyset \neq I(i) \neq [D]$ and we have just shown that I is antireflexive and symmetric, we can define a graph with vertices $[D]$ and edges given by I. Deleting edges in this graph equals discarding restrictions on the A_i, hence we consider only the case of spanning forests where the component trees are stars $S_1, \ldots, S_M$, each of size at least 2. Let i_l be the lowest-index vertex in the star S_l and $2 \leq D_l := |S_l| \leq D$. We get

$$
\begin{aligned}
& q \\
&= \mathbb{P}(f^{-1}(Y_v) \in A) \\
&\leq \mathbb{P}(f^{-1}(Y_v) \in \tilde{A}) \\
&\leq \prod_{i=1}^{D} \mathbb{P}(X_{(v,ind_v(i))} \in A_i) \\
&= \prod_{l=1}^{M} \prod_{i \in S_l} \mathbb{P}(X_{(v,ind_v(i))} \in A_i) \\
&\leq \prod_{l=1}^{M} \mathbb{P}(X_{(v,ind_v(i_l))} \in A_{i_l})[1 - \mathbb{P}(X_{(v,ind_v(i_l))} \in A_{i_l})]^{(D_l-1)} \\
&\leq \prod_{l=1}^{M} \frac{(D_l-1)^{(D_l-1)}}{D_l^{D_l}} \qquad \text{as } \sum_{l=1}^{M} D_l = D \\
&\leq \frac{(D-1)^{(D-1)}}{D^D} \qquad \text{applying (2.51a) recursively.}
\end{aligned}
$$

□

Another fact, using the notation from the proof of proposition 201, is that

$$\forall\, i \in [D] : \quad q = \int_{A_i} F_i(x)dx \leq \lambda(A_i) =: a_i\,,$$

whereupon we easily deduce that $0 < a_i < 1$. Using a measure-preserving transformation $T_i : [0,1] \to [0,1]$ with $T_i([0, a_i[) \mapsto A_i$, we write

$$\mathbb{P}(Y_v = 0) = \mathbb{P}(\forall\, i \in [D] : \quad X_{(v,ind_v(i))} \in T_i^{-1}(A_i))\,.$$

This amounts to, after retransformation in each coordinate, defining a Shearer-edge-factor by a subcube of $[0,1]^D$ of length $a_1 \times \ldots \times a_D$, which describes the 0-realization in the corresponding Y_v.

Proposition 202. *Let $G := (V, E)$ be the Cayley graph of a group Γ with respect to the finite generating set $S = \{a_1, \ldots, a_D\}$. Assume that no element in S has order 2. Then we can write $\mu_{G,p}$ as an edge-factor of Bernoulli rvs with parameter c, the solution of $q = c^D(1-c)^D$, iff $p \in \left[1 - \left(\frac{1}{2}\right)^{2D}, 1\right]$.*

Remark. Care has to be taken if one of the elements of S has order 2. Let $T := \{a \in S : a \text{ has order } 2\}$ and $E := |T|$. If $T \neq S$, then the construction works down to $1 - \frac{D^D(D-E)^{(D-E)}}{(2D-E)^{(2D-E)}}$, by giving each edge labelled by an element of T an arbitrary direction. Finally, if $T = S$, then we have a tree of degree D, in which case the construction works down to $1 - \frac{(D-1)^{(D-1)}}{D^D}$ and is isomorph to model 58, the optimal case in proposition 201.

On $\mathbb{Z}_{(k)}$, the k-fuzz of $\mathbb{Z}$, and for $k > 1$ we have the situation, that for $q \leq \frac{k^k}{(k+1)^{(k+1)}}$ we can explicitly construct Shearer's measure [MT12, section 4.2], but not an edge-factor. Whereas if $q \leq \frac{(k-1)^{2(k-1)}}{k^{2k}}$ we also have the construction as an edge-factor given in proposition 202.

Proof. For $p \geq 1 - \frac{D^D D^D}{(2D)^{(2D)}} = 1 - \left(\frac{1}{2}\right)$ define $X \sim \Pi_c^E$, with c as indicated above. For the index functions take the labels of the edges induced by the generating set S. The solution is given by

$$\forall\, v \in V : \quad Y_v := \prod_{a^{-1} \in S} (1 - X_{(v, ind_v(a))}) \prod_{a \in S} X_{(v, ind_v(a))} \,.$$

□

6.4.4 Tightness of a subclass of OVOEPs on $\mathbb{Z}^d$

We show that a certain subset of OVOEPs describes the tight asymptotics for $\Xi_\Lambda(p)$, as $\Lambda \nearrow \mathbb{Z}^d$. We use this tightness result to show the existence and analyticity of the infinite volume limit of the free energy of the polymer system in proposition 204. Contrast its proof with the proof of proposition 203 with the subadditive approach in [SS05, section 8.3], where the limit is shown to exist for the physical range of parameters. At the end of this section we discuss about the relation of this tightness result with the relevant intrinsic domination result from section 4.7.

Without loss of generality we give the statement only for $d = 2$. Proposition 203 should be easily generalizable to $\mathbb{Z}^d$ and other d-dimensional lattice graphs. I conjecture that a qualitative comparable statement holds on all infinite, locally finite quasi-transitive graphs with quasi-transitive parameters. The obstacle seems mostly notational (as in conjecture 63), especially in writing down the equivalent of $W_{(n,k,l)}$ (6.47a) in full generality.

Proposition 203. *Let $G := \mathbb{Z}^2$. For $n, k \in \mathbb{N}$ and $l \in \mathbb{N}_0$, define the set*

$$W_{(n,k,l)} := \left\{ (x,y) \in \mathbb{Z}^2 : \begin{array}{c} (0 \leq x < n \wedge 0 \leq y < k + l) \\ \vee \\ (x = n \wedge 0 \leq y < k) \end{array} \right\} . \tag{6.47a}$$

For $p \geq p_{sh}^{\mathbb{Z}^2}$ define the value

$$a(p) := \inf \{ \alpha_{W_{(n,k,l)}}^{(n,k)}(p) : \quad n, k \in \mathbb{N}, l \in \mathbb{N}_0 \} . \tag{6.47b}$$

The graph $G_N := G(W_{(N,N,0)})$ is the subgrid of side-length N with lower left corner at $(0,0)$. Then we have

$$\lim_{N \to \infty} -\frac{\log \Xi_{G_N}(p)}{N^2} = -\log a(p) < \infty \,. \tag{6.47c}$$

Proof of proposition 203. If $(n,k,l) \leq (\bar{n},\bar{k},\bar{l})$, then by (2.26a) we have

$$\alpha^{(n,k)}_{W_{(n,k,l)}}(p) \geq \alpha^{(\bar{n},\bar{k})}_{W_{(\bar{n},\bar{k},\bar{l})}}(p)\,.$$

Thus $a(p)$ is a limit in the sense of nets

$$a(p) = \lim_{n,k.l\to\infty} \alpha^{(n,k)}_{W_{(n,k,l)}}(p)\,.$$

All $\alpha^{(n,k)}_{W_{(n,k,l)}}$ are escaping and thus by (2.25b) $\alpha^{(n,k)}_{W_{(n,k,l)}}(p) \geq q$. Combining this with the monotonicity (2.26b) implies that $a(p)$ is monotone in p and that

$$a(p) \geq a(p^{\mathbb{Z}^2}_{sh}) \geq \log q^{\mathbb{Z}^2}_{sh} > -\infty\,. \tag{6.48}$$

Enumerate $W_{(N,N,0)} =: \{v_1,\dots,v_{N^2}\}$ by successive vertical lines:

$$(0,0),(0,1),\dots,(0,N-1),(1,0),(1,1),\dots,(N-1,N-1)\,.$$

Thus $W_i := \{v_1,\dots,v_{i-1}\} = W_{(n_i,k_i,l_i)}$ for suitable coefficients (n_i,k_i,l_i). This yields the identity

$$\Xi_{G_N}(p) = \prod_{i=1}^{N^2} \alpha^{v_i}_{W_i}(p) = \prod_{i=1}^{N^2} \alpha^{(n_i,k_i)}_{W_{(n_i,k_i,l_i)}}(p)\,.$$

On the one hand we have

$$\Xi_{G_N}(p) \geq a(p)^{N^2}\,. \tag{6.49}$$

On the other hand choose $\varepsilon > 0$. Then there exists a (n,k,l) such that $\alpha^{(n,k)}_{W_{(n,k,l)}}(p) \leq a(p)+\varepsilon$. We estimate roughly

$$\begin{aligned}\Xi_{G_N}(p) &= \Xi_{G(W_{(n,k,l)})}(p) \prod_{i=(n\vee(k+l+1))^2}^{N^2} \alpha^{(n_i,k_i)}_{W_{(n_i,k_i,l_i)}}(p)\\ &\leq \Xi_{G(W_{(n,k,l)})}(p)\,(a(p)+\varepsilon)^{N^2-(n\vee(k+l+1))^2-4N(k+l)}\,.\end{aligned}$$

Therefore

$$a(p)^{N^2} \leq \Xi_{G_N}(p) \leq \Xi_{G(W_{(n,k,l)})}(p)\,(a(p)+\varepsilon)^{N^2-(n\vee(k+l+1))^2-4N(k+l)}$$

and

$$\begin{aligned}\log a(p) \leq \frac{\log \Xi_{G_N}(p)}{N^2} \leq &\frac{\log \Xi_{G(W_{(n,k,l)})}(p)}{N^2}\\ &+ \frac{N^2-(n\vee(k+l+1))^2-4N(k+l)}{N^2}\log(a(p)+\varepsilon)\,.\end{aligned}$$

We first take the limit for $N\to\infty$ and then for $\varepsilon\to 0$, resulting in (6.47c). □

Proposition 204. *Let $\mathcal{P}$ be $\mathbb{Z}^2$, with incompatibility of nearest-neighbour edges. Then $F_{\mathcal{P}} : \mathbb{C}\to\mathbb{C}$ exists and is analytic on $D := \{z\in\mathbb{C} : |z| < q^{\mathbb{Z}^2}_{sh}\}$.*

Remark. Recall that section 5.2.3 asserts $-q_{sh}^{\mathbb{Z}^2}$ is the singularity of $F_{\mathbb{Z}^2}$ closest to 0.

Proof. A key point is *Montel's theory of normal families* [Mon27], for example in the form:

Lemma 205. *[SS05, proposition 8.12] Let D be a domain in $\mathbb{C}$. Let $S \subseteq D$ having at least one accumulation point in D. Let $0 < M < \infty$ and $(f_n)_{n\in\mathbb{N}}$ be analytic functions in D satisfying*

$$\forall\, n \in \mathbb{N}, z \in D: \quad \operatorname{Re} f_n(z) \leq M \tag{6.50a}$$

$$\forall\, z \in S: \quad \lim_{n\to\infty} f_n(z) \text{ exists and is finite}. \tag{6.50b}$$

We set $S := [-q_{sh}^{\mathbb{Z}^2}, 0]$ and $f_n := F_{W_{(n,n,0)}}$ (recall that $W_{(n,n,0)}$ is a square of side-length n). Then (6.50b) is just (6.47c). To show (6.50a) we write for $z \in D$:

$$\begin{aligned}
&\operatorname{Re} f_n(z)\\
&= \operatorname{Re}\left(-\frac{1}{n^2}\log \Xi_{W_{(n,n,0)}}(z)\right)\\
&= -\frac{1}{n^2}\operatorname{Re}\log \Xi_{W_{(n,n,0)}}(z)\\
&\leq -\frac{1}{n^2}\log \Xi_{W_{(n,n,0)}}(-|z|) && \text{by (5.8b)}\\
&= -\frac{1}{n^2}\log \Xi_{G_n}(1-|z|) && \text{switching terminology}\\
&\leq -\log a(1-|z|)\\
&\text{by (6.49)} && \leq -\log q_{sh}^{\mathbb{Z}^2} \text{by (6.48)}\\
&< \infty && \text{as } q_{sh}^{\mathbb{Z}^2} \in]0,1[\,.
\end{aligned}$$

□

We turn to $\sigma(\mu_{\mathbb{Z}^2,p})$. Proposition 112 asserts that $\mu_{\mathbb{Z}^2,p} \overset{st}{\geq} \Pi_{\mathbb{Z}^2}^{b(p)}$, where

$$b(p) := \alpha_{\mathbb{Z}^2\setminus\{(0,0),(1,0)\}}^{(0,0)}(p)\,.$$

Recall that

$$a(p) = \alpha_W^{(0,0)}(p)\,,$$

with $W := \{(x,y) \in \mathbb{Z}^2 : \quad x < 0 \vee (x = 0 \wedge y < 0)\}$. Joining the result (6.47c) implies that

$$b(p) \leq \sigma(\mu_{\mathbb{Z}^2,p}) \leq a(p)\,. \tag{6.51}$$

The upper bound follows from (6.47c) (use the monotone functions $f_W := \mathbb{I}_{\vec{1}_W}$).

Define the sets

$$\begin{aligned}
W_1 &:= \{(x,y) \in \mathbb{Z}^2 : \quad x < -1 \vee (x = -1 \wedge y < 0)\}\\
W_2 &:= \{(x,y) \in \mathbb{Z}^2 : \quad x > 1 \vee (x = 1 \wedge y > 0)\}\,.
\end{aligned}$$

That is W_2 is W_1 rotated 180°. This lets us write:

$$\begin{aligned}
& q_{sh}^{\mathbb{Z}^2} \\
&\leq b(p_{sh}^{\mathbb{Z}^2}) && \text{by escaping (2.25b)} \\
&\leq b(p) && \text{by monotonicity (2.26b)} \\
&< \alpha_{W_1 \uplus W_2}^{(0,0)}(p) && \text{by monotonicity (2.26a)} \\
&= 1 - \frac{q}{\alpha_{W_1 \setminus \{(-1,0)\}}^{(-1,0)}(p) \alpha_{W_2 \setminus \{(1,0)\}}^{(1,0)}(p)} && \text{by the fundamental identity (2.37)} \\
&= 1 - \frac{1}{a(p)^2} && \text{by translation invariance} \\
&\leq a(p) && \text{by monotonicity (2.26a).}
\end{aligned}$$

Therefore there is a substantial difference between $b(p)$ and $a(p)$. The relation $b(p) \leq 1 - \frac{1}{a(p)^2}$ also holds for higher dimensions. An immediate question is

Question 206. What is the correct value of $\sigma(\mu_{\mathbb{Z}^2,p})$? Is it one of the bounds in (6.51)?

If $\sigma(\mu_{\mathbb{Z}^2,p}) < a(p)$, then this would be a stark contrast to (2.56d) and (2.60d). My current non-understanding of the problem lets me not even formulate a reasonable conjecture. It seems to me though, that $a(p)$ misses some of the essential structure of $\mathbb{Z}^2$ and would require a two-dimensional equivalent of proposition (94) - something that I consider unlikely.

For $p < p_{sh}^{\mathbb{Z}^2}$, the following related and more generally formulated question arises:

Question 207. Given a Cayley graph G on $\mathbb{Z}^d$ generated by S and knowing that $p_{dom}^G = p_{sh}^G$ as well as the upper bound on p_{sh}^G from theorem 187: Can we use $\prec_{\vec{a}}$ to construct an $X \in \mathcal{C}_G^{\text{strong}}(p)$ such that $\sigma(X) = 0$? This is, without going through Shearer's measure and using lemma 101?

6.5 Known rigorous bounds and exact solutions

This section compares known bounds on $q_{sh}^{\mathbb{Z}^2}$ for the integer lattice $\mathbb{Z}^2$. We enumerate both upper bounds from the pruned tree techniques in section 6.4.1, numerical estimates from various sources in the literature as well as lower bounds from the different criteria in sections 2.2.6 and 2.5.2 and the techniques developed in chapter 5. We outline the steps to calculate higher depth conditions derived from tree-operator bounds in practise in section 6.5.1 and give the bounds themselves in section 6.5.2.

6.5.1 Calculating tree-operator bounds

An application of the theory of tree-operators in chapter 5 boils down to an explicit enumeration of a finite family of rooted trees. This section describes how to the calculations for higher depth approximation bounds work. The depth one approximations bounds are given in sections 5.2.

We work in the setting of proposition 142. Thus we have a polymer system $\mathcal{P}$ with incompatibility relation $\approx$, a partition scheme S and a depth $k \in \mathbb{N}$. We have an approximation of the form (5.37a), leading to a series of the type (5.35c). The convergence criterion for this series is (5.35b), for a suitable label space $\mathcal{L}$ and $\vec{\rho}, \vec{\mu} \in]0, \infty[^{\mathcal{L}}$. Our aim is the calculation of this criterion. Therefore we calculate $\forall\, l \in \mathcal{L}$ the term

$$[T_{\vec{\rho}}(\vec{\mu})]_l = \sum_{t \in \mathfrak{T}_k^{\leq}} \frac{1}{\mathfrak{p}(t)} \sum_{\vec{\lambda} \in \{l\} \times \mathcal{L}^{|W(t)|}} c_t(\vec{\lambda}) \prod_{\substack{v \in L_i(t) \\ 0 \leq i \leq k-1}} \rho_{\lambda_v} \prod_{w \in L_k(t)} \mu_{\lambda_v} \,. \tag{6.52}$$

Recall that the $(c_t)_{t \in \mathfrak{T}_k^{\leq}}$ are indicator functions invariant under rooted automorphisms of the tree, that is invariant under every reordering of the children. For a given tree t, the number of such automorphisms is $\mathfrak{p}(t)$. We have a projection function $\downarrow : \mathcal{L} \to \mathcal{P}$. This allows us to rewrite the criterion as a sum over a finite family F of rooted, $\mathcal{P}$-labelled trees F, with labelling functions $\downarrow \circ l_t : V(t) \to \mathcal{P}$:

$$[T_{\vec{\rho}}(\vec{\mu})]_l = \sum_{t \in F} \left[\left(\prod_{i=0}^{k-1} \prod_{v \in L_k(t)} \rho_{\downarrow(l_t(v))} \right) \prod_{v \in L_k(t)} \mu_{\downarrow(l_t(v))} \right] . \tag{6.53}$$

Hence the problem is reduced to the description of $\mathcal{L}$, $\downarrow$ and the family F. The description of $\mathcal{L}$, $\downarrow$ and l_t translates into constraints on the $\mathcal{P}$-valued labels t can carry in order to belong to F.

We also need total orders on $\mathcal{P}$: namely a global one $\prec$, local ones $\prec_\gamma$, for $\gamma \in \mathcal{P}$ and path orders $\prec_P$, for $P \in \mathrm{W}((\mathcal{P}, \approx))$. To reduce the number of these additional parameters we can eliminate the path orders by expressing them by the local orders:

$$\forall\, \overline{\ldots\gamma} := P \in \mathrm{W}(\mathcal{P}) \quad \xi \prec_P \bar{\xi} \Leftrightarrow \xi \prec_\gamma \bar{\xi} \,. \tag{6.54a}$$

All our example graphs $(\mathcal{P}, \approx)$ (with loops) are *translation-invariant*. This lets us express the local orders by translation to the origin

$$\forall\, \gamma \in \mathcal{P} : \quad \xi \prec_\gamma \bar{\xi} \Leftrightarrow (\xi - \gamma) \prec (\bar{\xi} - \gamma) \,. \tag{6.54b}$$

Combining (6.54b) and (6.54a) we get

$$\forall\, \overline{\ldots\gamma} := P \in \mathrm{W}(\mathcal{P}) \quad \xi \prec_P \bar{\xi} \Leftrightarrow \xi \prec_\gamma \bar{\xi} \Leftrightarrow (\xi - \gamma) \prec (\bar{\xi} - \gamma) \,. \tag{6.54c}$$

In view of the partition of the complement of a spanning tree of a cluster (5.33) the family F must ensure, that for every $t \in F$ and every ordering of the vertices of t all edges $e \in V(t)^2 \setminus E(t)$ are *conflicting*, that is $e \in \mathcal{C}_S(t)$.

A preliminary example is

Example 208 (Kotecký & Preiss for depth k)**.** We approximate S by only keeping compatibility restrictions between the polymer labels of a vertex and its parent. That is, for a tree t with root o to be in F we demand, that

$$\mathrm{depth}(t) \leq k \tag{6.55a}$$

$$\forall\, v \in V(t) \setminus \{o\} : \quad \xi_v \approx \xi_{\mathfrak{p}(v)} \,. \tag{6.55b}$$

The family F given in example 208 is not finite. Nevertheless the condition (6.55b) only depends on adjacent levels and reduces to an iterated depth one approximation. This is just the classic Kotecký & Preiss condition. See also section 5.7.1. By the nature of the cluster expansion (6.55) has to hold for every singleton tree in the partition with respect to every partition scheme S of a cluster.

Example 209 (Penrose with cousins of depth k)**.** We approximate *Pen* by only keeping *cousin* restrictions, that is restriction (5.42a). That is, in addition to (6.55), we demand, that

$$\forall\, i \in [k] : \quad \mathrm{supp}(\vec{\xi}_{L_i(t)}) \text{ is a compatible subset of } \mathcal{P}\,. \tag{6.56}$$

Demand (6.56) is equivalent to $l_t(L_i(t))$ being an independent set in $(\mathcal{P}, \approx)$. Because the condition spans the whole level of a tree a depth one approximation as in (5.15b) is strictly worse than a depth two approximation and so on. Improving the approximation from example 209 we have

Example 210 (Penrose both of depth k)**.** We approximate *PenM* from section 5.5.4 instead of *Pen*. To (6.55) and (6.56) we add the *uncle restrictions*

$$\forall\, i \in [k], \forall\, v \in L_i(t), w \in L_{i-i}(t) : \quad \xi_{\mathfrak{p}(v)} \prec_{\xi_v} \xi_w \;\Rightarrow\; \xi_w \not\approx \xi_v\,. \tag{6.57}$$

Demand (6.57) comes from (5.50), ensuring that all such (v, w) are conflicting. We can apply the same improvement to *Ret* as to *Pen* in section 5.5.4 and the longer history improvement from section 5.6.5.

Example 211 (Returning of depth k and with history size m)**.** Let t be a finite tree rooted at o with polymer labels. We apply the status labelling and class construction from algorithm 153 to t. Then $t \in F$ iff (6.55) holds and

$$\forall\, \text{class } C : \quad \mathrm{supp}(\vec{\xi}_C) \text{ is a compatible subset of } \mathcal{P} \tag{6.58a}$$

$$\forall\, \text{class } C \neq \{o\} : \forall\, v \in C, w \in \mathfrak{p}(C) : \quad \xi_{\mathfrak{p}(v)} \prec_{\xi_v} \xi_w \;\Rightarrow\; \xi_w \not\approx \xi_v \tag{6.58b}$$

$$\forall\, v \in V(t) : \quad \xi_v \notin h(v)\,, \tag{6.58c}$$

where $h : V \to \mathrm{SAW}((\mathcal{P}, \approx))$ is a length m SAW in $\mathcal{P} \setminus \mathrm{supp}(\vec{\xi}_V)$ ending at a neighbour of ξ_o in $(\mathcal{P}, \approx)$ and is extended every time there is a $\mathbf{D}$ transition on $P(o, v)$.

Here (6.58a) and (6.58b) are the same as in example 210, but softened to apply only within each class. This is countered by (6.58c), which has been the primary reason for the introduction of the classes.

We have implemented algorithms to construct the above examples of finite families and several variations thereof in *Standard ML* [RMRD97], in particular for the implementations SML/NJ, Moscow ML and MLton. The evaluation of the resulting polynomials is done in Python and Sage.

6.5.2 The planar grid $\mathbb{Z}^2$

We list all bounds for $q_{sh}^{\mathbb{Z}^2}$ known to us in table 6.5.2.

Bound on $q_{sh}^{\mathbb{Z}^2}$	Source of bound
$\leq \frac{\sqrt{2}^{\sqrt{2}}}{(\sqrt{2}+1)^{(\sqrt{2}+1)}} \sim 0.1944$	Dimensional pruned SAW (6.39) with $m = 2$.
$\leq \frac{2^2}{3^3} = \frac{4}{27} \sim 0.1481$	Vectorized pruned SAW tree (6.31).
≤ 0.1269	Stacked pruned SAW tree (6.43).
~ 0.1193	Exact solution by transfer matrix techniques by Todo [Tod99].
~ 0.1193	Exact solution by series expansion by Guttman [Gut87].
≥ 0.1071	Tree-operator bound as described in example 211 with $k = 2$ and $s = 1$.
$\geq \frac{3^3}{4^4} \sim \frac{27}{256} \sim 0.10545$	Tree-operator bound as described in example 211 with $k = 1$ and $s = 1$. We optimize (6.60), which is a special case of (2.42a). .
≥ 0.0994	Tree-operator bound as described in example 209 with $k = 3$.
≥ 0.0979	Tree-operator bound as described in example 209 with $k = 2$, as solution of $p(\rho, \mu) \leq \mu$, The polynomial $p(\rho, \mu)$ optimized here is given in (6.61). There are already 12857 trees in the finite family, each adding a monomial term.
≥ 0.0955	After (2.46a) and (5.15b) from [FP07]. We optimize (6.59).
$\geq \frac{4^4}{5^5} \sim 0.0819$	The classic Dobrushin-condition (5.13a) or the symmetric LLL (2.41) by Erdös & Lovász.

Table 6.1: Known bounds for the planar grid, that is $q_{sh}^{\mathbb{Z}^2}$. The left column lists the bounds in the following order: upper bounds (necessary conditions for $q \leq q_{sh}^{G}$), exact solutions and then lower bounds (sufficient conditions for $q \leq q_{sh}^{G}$). All results have been rounded to the 5^{th} digit, for easier comparison. The right column explains the origin of the bound/solution.

In the case of depth $k = 1$ bounds, as in (6.60), the condition always factorizes into the form $\rho \leq \max(f(\mu))$, while in the higher depth cases, as in (6.61), one has to search for solutions numerically. An example of a graphical search is given in figure 6.1. Following are some of the polynomials, or their transformations, used in table 6.5.2.

$$q_{sh}^{\mathbb{Z}^2} \geq \max\left\{\mu : \quad \frac{\mu}{\mu + \Xi_{\mathcal{I}^\star((0,0))}(\mu)} = \frac{\mu}{\mu + (1+\mu)^4}\right\}. \tag{6.59}$$

$$q_{sh}^{\mathbb{Z}^2} \geq \max\left\{\mu : \quad \frac{\mu}{(1+\mu)\Xi_{\mathcal{I}^\star((0,0))\setminus(0,1)}(\mu)} = \frac{\mu}{(1+\mu)^4}\right\}. \tag{6.60}$$

$$\begin{aligned} p(\rho,\mu) := {} & 64\rho^5\mu^9 + 400\rho^5\mu^8 + 1088\rho^5\mu^7 + 48\rho^4\mu^8 + 1704\rho^5\mu^6 + 304\rho^4\mu^7 \\ & + 1716\rho^5\mu^5 + 828\rho^4\mu^6 + 4\rho^3\mu^7 + 1168\rho^5\mu^4 + 1284\rho^4\mu^5 + 42\rho^3\mu^6 \\ & + 528\rho^5\mu^3 + 1260\rho^4\mu^4 + 152\rho^3\mu^5 + 144\rho^5\mu^2 + 808\rho^4\mu^3 + 278\rho^3\mu^4 \\ & + 20\rho^5\mu + 312\rho^4\mu^2 + 296\rho^3\mu^3 + 5\rho^2\mu^4 + \rho^5 + 60\rho^4\mu + 192\rho^3\mu^2 \\ & + 20\rho^2\mu^3 + 4\rho^4 + 60\rho^3\mu + 30\rho^2\mu^2 + 6\rho^3 + 25\rho^2\mu + 5\rho^2 + \rho\,. \end{aligned} \tag{6.61}$$

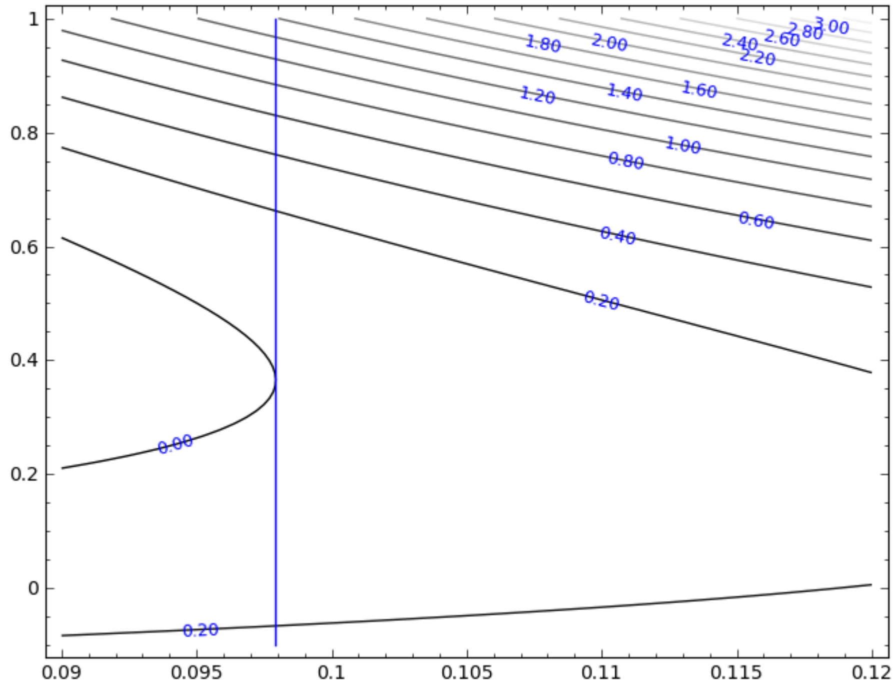

Figure 6.1: (Colour online) The contour plot of $p(\rho, \mu) - \mu$ from (6.61) in the critical parameter regime. The abscissa is ρ and the ordinate is μ. The vertical line has abscissa 0.0979 and represents the maximum $\rho^\star$ for which there is a $\mu^\star$ such that $p(\rho^\star, \mu^\star) \leq \mu^\star$. We recall that $p(\rho, \mu)$ is the polynomial of the generating function of the finite family of trees from the greedy cousin approximation from example 209 with depth $k = 2$ for the planar grid $\mathbb{Z}^2$.

I looked into the abyss of my mind;
and the abyss stared back.

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